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Numerical integration of shell equations using the field method

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NUMERICAL INTEGRATION OF SHELL EQUATIONS USING THE FIELD METHOD

BY

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A

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ABSTRACT

In this work field equations are introduced to numerically solve the six linear differential equations which determine the displacements and stress resultants for thin elastic shells with axially symmetric loadings. With the applications of the field equations to the analysis of symmetric shells, the two-point boundary-value problem is formulated in terms of twelve first-order ordinary differential equations with boundary conditions at only one point. These equations are solved by using a forth-order Runge-Kutta integration formula.

Three problems whose solutions are known have been evaluated to check the accuracy of the field equation method. These are:

1. Simply-supported, thin circular cylindrical shell of finite length with a uniform internal pressure.
2. Simply-supported, thin circular cylindrical shell of finite length, under a radial line load distributed around the circumference at the center section.
3. Thin circular cylindrical shell with both ends fixed and a uniform internal pressure.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

(Listed in order of appearance)

n - order of differential equation

x, y, z - rectangular cartesian coordinates

θ - longitude of a meridian

s - arclength measured from the vertex

r - radius of a cross-sectional circle of a surface of revolution

A, B - metric coefficients for orthogonal coordinates on a surface

e, f, g - coefficients of second fundamental form of a surface

M_s, M_θ - bending moments in a shell

$M_{s\theta}, M_{\theta s}$ - twisting moments in a shell

N_s, N_θ - resultant tensile force in a shell

Q_s, Q_θ - transverse shears in a shell

p - load per unit area of the coordinate (middle) surface

L_s, L_θ - external moments applied to a shell

r_1, r_2 - principal raddi of curvature of the middle surface of the shell

$K = \frac{1}{r_1}$ - principal curvature of surface

E - Young's modulus

h - thickness of shell

ν - Poisson's ratio

$\varepsilon_s, \varepsilon_\theta$ - strain components of the middle surface of the shell

K_s, K_θ - quantities related to changes of curvature

N, M, Q - nondimensionalized forms of N_s, M_s, Q_s

u - displacement component in the s-direction

W - displacement component in the $\hat{n}$-direction

$\hat{n}$ - principal normal

f_{uW}, f_{uN}, f_{uM}, f_u, $f_{W'W}$, $f_{W'N}$, $f_{W'M}$, $f_{W'}$, f_{QW}, f_{QN}, f_{QM}, and f_Q - field functions

P - radial line load

a - radius of the cylinder

ℓ - length of the shell

D - flexural rigidity of the shell

β - as defined in context

α - as defined in context

r_f - radius of the shell at the base

LIST OF TABLES

I. INTRODUCTION

In order to solve for the stress distribution in complex shells, approximations must be resorted to which require many lengthy and tedious calculations. The numerical integration of two-point boundary-value problems entails difficulties if the problem is "unstable"; that is, if an unwanted homogeneous solution is divergent to such a degree that the alloted register capacity of the digital computer tends to be used up in representing the unwanted rather than the wanted solution. A technique for converting certain ordinary differential equations whose solution is exponential in character to a set of first-order differential equations that do not possess this property has been discussed by R. E. Miller (1)*. This method is known as the field method.

With the advent of high-speed computing systems and with the help of the field method, it is possible to solve with good accuracy an even-order system of ordinary linear differential equations which have two-point boundary conditions. The field method converts an ordinary differential equation of order 2n (or a general even-ordered system) with boundary conditions at the end points a and b into a set of n(n + 1) first-order equations that must be integrated from a to b and a second set of n first-order equations that are then integrated backwards from b to a.

The advantages in applying the field method to the

*Numbers in parentheses refer to the list of references at the end of the thesis.

numerical solution of two-point boundary-value problems are:

(1) More accurate results can be obtained than by using finite-difference-type methods with a comparable amount of computer time and

(2) Ability to handle problems in which the end points are widely separated.

Problems in which the end points are widely separated are not well suited for other numerical integration procedures because of the rapid build-up of the exponential portion of the solution.

II. REVIEW OF LITERATURE

The shell of revolution is an important structural element, and the literature devoted to its analysis is extensive. As far as axisymmetric deformation is concerned, many methods have been applied to obtain solutions of the H. Reissner-Meissner equations of the linear bending theory of shells of revolution. For example, Naghdi and DeSilva (2) use asymptotic integration; Wilson and Spier (3) apply an iterative scheme with a direct-elimination method; Galletly, et al (4) find the solution for an ellipsoidal shell of revolution by both the finite-difference and the direct-integration method; and Radkowski, et al (5) and Sepetoski et al (6) utilize the finite-difference technique. A number of additional references which deal with the solution of the H. Reissner-Meissner equations can be found in the papers cited. Among the papers which employ numerical analysis, two different methods of solution of the boundary-value problem of deformation of shells must be recognized. They are the direct-integration (4) and the finite-difference approach, (4) through (6).

While the direct-integration approach has certain important advantages, it also has a serious disadvantage. When the length of the shell is increased, a loss of accuracy invariably results. This phenomenon was clearly pointed out in (6). The loss of accuracy does not result from accumulative errors in integration, but it is caused by the subtraction of almost equal numbers in the process of determination of the unknown boundary values. It follows that for every set of geometric and material parameters of the shell there is a critical length beyond which the solution loses all accuracy. The advantage of the finite-difference approach over direct integration is that it can avoid such a loss of accuracy.

A numerical method of solution which combines the advantages of both the direct-integration and finite-difference approach is presented in (7) for the analysis of symmetric shells. This method eliminates the loss of accuracy encountered in the usual application of the direct-integration approach to the analysis of shells.

An integration-suppression method was discussed in (8) for ordinary differential equations whose solution are exponential in character. Its solution consists of the following steps.

1. A set of points of suppression is introduced in the finite shell.
2. The numerical integration is carried out through the first interval, thus obtaining partial solutions.
3. The partial solutions are combined to satisfy a pair of independent sets of arbitrary conditions at the points of suppression.
4. These two solutions are propagated through the next interval and step 3 is applied at the next point of suppression.
5. This procedure is continued until the actual

conditions at the end can be applied to determine the constants of integration.

This method requires about 50% less computation time on a digital computer than the method discussed in (7).

The methods discussed in (7) and (8) require that the shells be divided into segments or that suppression points be selected. This generally requires considerable judgement and experience to obtain good results efficiently. Use of the field equations eliminates the necessity of dividing the shell or selecting suppression points.

III. FIELD EQUATIONS FOR SHELLS OF REVOLUTION

The shell equations*

$$\frac{du}{dx} = \frac{-\nu r'}{r} u + (K + \frac{\nu}{r_2}) W + \frac{(1 - \nu^2)}{h} N \tag{1}$$

$$\frac{dW}{dx} = W' \tag{2}$$

$$\frac{dW'}{dx} = - \frac{\nu r'}{r} W' - \frac{12(1 - \nu^2)}{h^3} M \tag{3}$$

$$\frac{dN}{dx} = \frac{hr'^2}{r^2} u - \frac{hr'}{rr_2} W - \frac{r'(1 - \nu)}{r} N + KQ \tag{4}$$

$$\frac{dM}{dx} = - \frac{h^3 r'^2}{12r^2} W' - \frac{r'(1 - \nu)}{r} M + Q \tag{5}$$

$$\frac{dQ}{dx} = - \frac{hr'}{rr_2} u + \frac{h}{r_2^2} W - (K + \frac{\nu}{r_2}) N - \frac{r'}{r} Q - P_z \tag{6}$$

and the boundary conditions for a closed shell of revolution

$$u(0) = 0$$

$$Q(0) = \text{known}$$

$$W'(0) = 0$$

plus three conditions at the base of the shell constitute a two-point boundary-value problem. Let the relations between the variables whose initial values are known and those whose initial values are unknown be expressed as follows:**

$$u = f_{uW}W + f_{uN}N + f_{uM}M + f_u \tag{7}$$

$$W' = f_{W'W}W + f_{W'N}N + f_{W'M}M + f_{W'} \tag{8}$$

$$Q = f_{QW}W + f_{QN}N + f_{QM}M + f_Q \tag{9}$$

---

*These equations are derived in Appendix-A.

**These equations are equivalent to equation (5) in reference (1).

Equations (7), (8) and (9) express the three functions u, W' and Q for which initial values are given as linear functions of W, N and M. Initial values for W, N and M are not known. In Equations (7), (8) and (9), the f_{uW}, f_{uN}, f_{uM}, f_u, $f_{W'W}$, $f_{W'N}$, $f_{W'M}$, $f_{W'}$, f_{QW}, f_{QN}, f_{QM}, and f_Q are unknown functions of x (the field functions) which are to be determined.

Differentiating Equation (7) and using Equation (1), the following set of equations is obtained.*

$$f'_{uW}{}^{**} = -f_{uW}[f_{W'W} + \frac{\nu r'}{r}] - f_{uN}[\frac{hr'^2}{r^2} f_{uW} - \frac{hr'}{rr_2} + Kf_{QW}]$$
$$-f_{uM}[\frac{-h^3r'^2}{12r^2} f_{W'W} + f_{QW}] + [K + \frac{\nu}{r_2}] \quad (10)$$

$$f'_{uN} = -f_{uW}f_{W'N} - f_{uN}[\frac{hr'^2}{r^2} f_{uN} + Kf_{QN} - \frac{r'}{r}(1 - 2\nu)$$
$$-f_{uM}[\frac{h^3r'^2}{12r^2} f_{W'N} + f_{QN}] + [\frac{1 - \nu^2}{h}] \quad (11)$$

$$f'_{uM} = -f_{uW}f_{W'M} - f_{uN}[\frac{hr'^2}{r^2} f_{uM} + Kf_{QM}]$$
$$-f_{uM}\left[\frac{-h^3r'^2}{12r^2} f_{W'M} + f_{QM} - \frac{r'}{r}(1 - 2\nu)\right] \quad (12)$$

and

$$f'_u = -f_{uW}f_{W'} - f_{uN}\left[\frac{hr'^2}{r^2} f_u + Kf_Q\right]$$
$$-f_{uM}\left[\frac{-h^3r'^2}{12r^2} f_{W'} + f_Q\right] - \frac{\nu r'}{r} f_u \quad (13)$$

Likewise, differentiating Equations (8) and (9) and using Equations (3) and (6), respectively, the following results

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*These equations are equivalent to equation (7) in reference (1).

**Primes are used to denote derivatives with respect to x, i.e., $f'_{uW} = df_{uW}/dx$, etc.

are obtained:

$$f'_{W'W} = -f_{W'W}\left[f_{W'W} + \frac{\nu r'}{r}\right] - f_{W'N}\left[\frac{hr'^2}{r^2} f_{uW} + Kf_{QW} - \frac{hr'}{rr_2}\right]$$

$$-f_{W'M}\left[\frac{-h^3r'^2}{12r^2} f_{W'W} + f_{QW}\right] \qquad (14)$$

$$f'_{W'N} = -f_{W'W}\, f_{W'N} - f_{W'N}\left[\frac{hr'^2}{r^2} f_{uN} + Kf_{QN} - \frac{r'}{r}(1 - 2\nu)\right]$$

$$-f_{W'M}\left[\frac{-h^3r'^2}{12r^2} f_{W'N} + f_{QN}\right] \qquad (15)$$

$$f'_{W'M} = -f_{W'W}\, f_{W'M} - f_{W'N}\left[\frac{hr'^2}{r^2} f_{uM} + K f_{QM}\right]$$

$$- f_{W'M}\left[\frac{-h^3r'^2}{12r^2} f_{W'M} + f_{QM} - \frac{r'}{r}(1 - 2\nu)\right]$$

$$\frac{-12(1 - \nu^2)}{h^3} \qquad (16)$$

$$f'_{W'} = -f_{W'W}\, f_{W'} - f_{W'N}\left[\frac{hr'^2}{r^2} f_u + K f_Q\right]$$

$$-f_{W'M}\left[\frac{-h^3r'^2}{12r^2} f_W + f_Q\right] - \frac{\nu r'}{r} f_{W'} \qquad (17)$$

$$f'_{QW} = -f_{QW}\left[f_{W'W} + \frac{r'}{r}\right]$$

$$-f_{QN}\left[\frac{hr'^2}{r^2} f_{uW} + K f_{QW} - \frac{hr'}{rr_2}\right]$$

$$-f_{QM}\left[\frac{-h^3r'^2}{12r^2} f_{W'W} + f_{QW}\right] - \frac{hr'}{rr_2} f_{uW} + \frac{h}{r_2^{\,2}} \qquad (18)$$

$$f'_{QN} = -f_{QW}\, f_{W'N} - f_{QN}\left[\frac{hr'^2}{r^2} f_{uN} + K f_{QN} + \frac{r'}{r}\nu\right]$$

$$-f_{QM}\left[\frac{-h^3r'^2}{12r^2} f_{W'N} + f_{QN}\right] - \frac{hr'}{rr_2} f_{uN} - (K + \frac{\nu}{r_2}) \qquad (19)$$

$$f'_{QM} = -f_{QW}\, f_{W'M} - f_{QN}\left[\frac{hr'^2}{r^2} f_{uM} + K\, f_{QM}\right] - f_{QM}\left[\frac{-h^3 r'^2}{12r^2} f_{W'M} + f_{QM} + \frac{r'}{r}\,\nu\right] - \frac{hr'}{rr_2} f_{W'M} \tag{20}$$

$$f'_{Q} = -f_{QW}\, f_{W'} - f_{QN}\left[\frac{hr'^2}{r^2} f_{u} + K\, f_{Q}\right] - f_{QM}\left[\frac{-h^3 r'^2}{12r^2} f_{W'} + f_{Q}\right] - \frac{hr'}{rr_2} f_{W'} - \frac{r'}{r} f_Q - P_z \tag{21}$$

Equations (10) through (21) are known as the field function equations for axisymmetric shells. The field functions f_{uW}, f_{uN}, f_{uM}, f_u, $f_{W'W}$, $f_{W'N}$, $f_{W'M}$, $f_{W'}$, f_{QW}, f_{QN}, f_{QM}, and f_Q, as well as their first derivatives, are involved in Equations (10) through (21), but only one derivative term is present in each of the field equations.

At x = 0, the values of u, W', and Q are known. They cannot depend on W, N, and M, the variables whose values are not known. Therefore, for Equations (7), (8), and (9) to be valid at x = 0

$$\begin{aligned} &f_{uW}(0) = f_{uN}(0) = f_{uM}(0) = 0 && f_u(0) = u(0) \\ &f_{W'W}(0) = f_{W'N}(0) = f_{W'M}(0) = (0) && f_{W'}(0) = W'(0) \\ &f_{QW}(0) = f_{QN}(0) = f_{QM}(0) = 0 && f_Q(0) = Q(0) \end{aligned} \tag{22}$$

The initial values of all the field functions are known. Thus, the field functions can be determined by direct integration on the interval $[0 \leq x \leq b]$. At $x = b$, the three given boundary conditions and Equations (7), (8), and (9) yield six equations which determine the six variables u, W, W', N, M, and Q. All the variables in the shell equations are now known at one point, $x = b$. The shell equations could be integrated backwards from $x = b$ to $x = 0$ to give the desired results. However, rather than integrate all six equations, Equations (7), (8) and (9) can be substituted into Equations (1) through (6) to give the equations

$$\frac{dW}{dx} = f_{W'W}\, W + f_{W'N}\, N + f_{W'M}\, M + f_{W'} \tag{23}$$

$$\frac{dN}{dx} = \frac{hr'^2}{r^2}\left(f_{uW}\, W + f_{uN}\, N + f_{uM}\, M + f_u\right) - \frac{hr'}{rr_2}\, W$$
$$\frac{-r'(1-\nu)}{r}\, N + K\left(f_{QW}\, W + f_{QN}\, N + f_{QM}\, M + f_Q\right) \tag{24}$$

$$\frac{dM}{dx} = \frac{-h^3 r'^2}{12r^2}\left(f_{W'W}\, W + f_{W'N}\, N + f_{W'M}\, M + f_{W'}\right)$$
$$+ \left(f_{QW}\, W + f_{QN}\, n + f_{QM}\, m + f_Q\right)$$
$$- \frac{r'}{r}(1-\nu)\, M \tag{25}$$

for determining W, N, and M by integrating from $x = b$ to $x = 0$. Equations (7), (8), and (9) can be used to calculate u, W', and Q after W, N, and M are determined.

For a circular cylindrical shell, the radius of curvature r_1 is infinite and r_2 is equal to the radius of the cylinder. Thus

$$r_2 = a \qquad\qquad K = \frac{1}{r_1} = 0$$

and $\qquad r = a, \qquad r' = 0$

By substituting the values of r, r_1 and r_2 in the field equations (10) through (21), and also in Equations (23) through (25), the following field equations for circular cylindrical shells are obtained.

$$f'_{uW} = -f_{uW}\, f_{W'W} - f_{uM}\, f_{QW} + \frac{\nu}{a} \tag{26}$$

$$f'_{uN} = -f_{uW}\, f_{W'N} - f_{uM}\, f_{QN} + \frac{1 - \nu^2}{h} \tag{27}$$

$$f'_{uM} = -f_{uW}\, f_{W'M} - f_{uM}\, f_{QM} \tag{28}$$

$$f'_{u} = -f_{uW}\, f_{W'} - f_{uM}\, f_{Q} \tag{29}$$

$$f'_{W'W} = -f_{W'W}^{\;2} - f_{W'M}\, f_{QW} \tag{30}$$

$$f'_{W'N} = -f_{W'W}\, f_{W'N} - f_{W'N}\, f_{QN} \tag{31}$$

$$f'_{W'M} = -f_{W'W}\, f_{W'M} - f_{W'M}\, f_{QM} - \frac{12(1 - \nu^2)}{h^3} \tag{32}$$

$$f'_{W'} = -f_{W'W}\, f_{W'} - f_{W'M}\, f_{Q} \tag{33}$$

$$f'_{QW} = -f_{QW}\, f_{W'W} - f_{QW}\, f_{QM} + \frac{h}{a^2} \tag{34}$$

$$f'_{QN} = -f_{QW}\, f_{W'N} - f_{QN}\, f_{QM} - \frac{\nu}{a} \tag{35}$$

$$f'_{QM} = -f_{W'M}\, f_{QW} - f_{QM}^{\;2} \tag{36}$$

$$f'_{Q} = -f_{QW}\, f_{W'} - f_{QM}\, f_{Q} - P_z \tag{37}$$

$$\frac{dW}{dx} = f_{W'W}\, W + f_{W'N}\, N + f_{W'M}\, M + f_{W'} \tag{38}$$

$$\frac{dN}{dx} = 0 \tag{39}$$

$$\frac{dM}{dx} = f_{QW}\, W + f_{QN}\, N + f_{QM}\, M + f_Q \tag{40}$$

In the next chapter Equations (26) through (40) will be used to determine the displacements and stress resultants in cylindrical shells subjected to various loadings.

IV. EXAMPLE PROBLEMS

Problem-1

Simply supported, thin circular cylindrical shell of finite length with a uniform internal pressure.

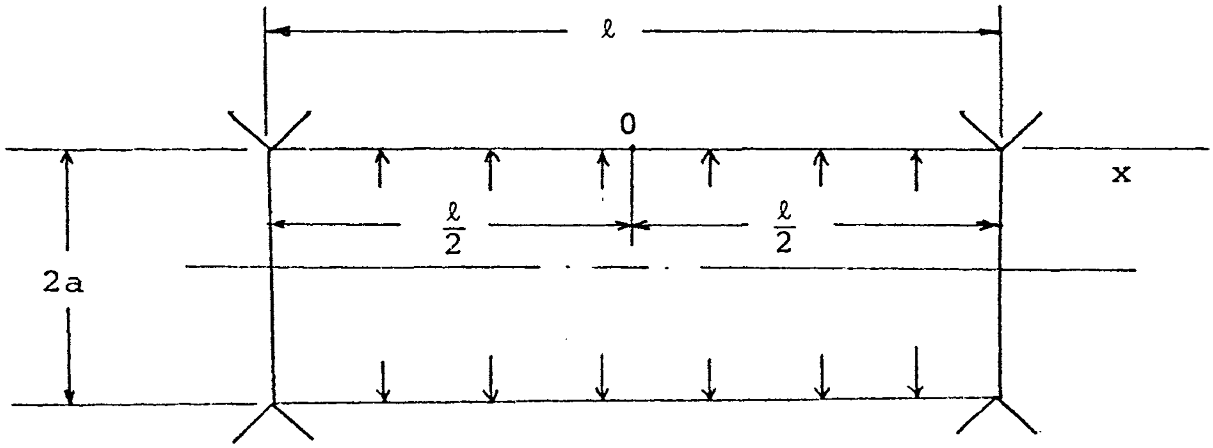


Figure 1. Cylindrical shell of finite length with both ends simply-supported and a uniform internal pressure.

The boundary conditions at the center section are:

$$(u)_{x=0} = 0$$

$$(W')_{x=0} = 0$$

$$(Q)_{x=0} = 0$$

and at the supported ends

$$(W)_{x=\ell/2} = 0$$

$$(N)_{x=\ell/2} = 0$$

$$(M)_{x=\ell/2} = 0$$

$$E_{st} = 30 \times 10^6 \ \#/\text{in.}^2$$

$$\nu = 0.25$$

$$h/a = 0.01$$

$$p = 500 \ \#/\text{in.}^2$$

$$a = 2 \text{ in.}$$

Using all the known values in the field equations (26) through (40), the following results are obtained:

$$f'_{uW} = 0.250 - f_{uW}\, f_{W'W} - f_{uM}\, f_{QW} \tag{41}$$

$$f'_{uN} = 93.75 - f_{uW}\, f_{W'N} - f_{uM}\, f_{QN} \tag{42}$$

$$f'_{uM} = -\, f_{uW}\, f_{W'M} - f_{uM}\, f_{QM} \tag{43}$$

$$f'_{u} = -\, f_{uW}\, f_{W'} - f_{uM}\, f_{Q} \tag{44}$$

$$f'_{W'W} = -\, f_{W'W}^{\,2} - f_{W'M}\, f_{QW} \tag{45}$$

$$f'_{W'N} = -\, f_{W'W}\, f_{W'N} - f_{W'M}\, f_{QN} \tag{46}$$

$$f'_{W'M} = -11250000.0 - f_{W'W}\, f_{W'M} - f_{W'M}\, f_{QM} \tag{47}$$

$$f'_{W'} = -f_{W'W}\, f_{W'} - f_{W'M}\, f_{Q} \tag{48}$$

$$f'_{QW} = 0.010 - f_{QW}\, f_{W'W} - f_{QW}\, f_{QM} \tag{49}$$

$$f'_{QN} = -0.250 - f_{QW}\, f_{W'N} - f_{QN}\, f_{QM} \tag{50}$$

$$f'_{QM} = -\, f_{W'M}\, f_{QW} - f_{QM}^{\,2} \tag{51}$$

$$f'_{Q} = -0.00001667 - f_{QW}\, f_{W'} - f_{QM}\, f_{Q} \tag{52}$$

$$\frac{dW}{dx} = f_{W'W}\, W + f_{W'N}\, N + f_{W'M}\, M + f_{W'} \tag{53}$$

$$\frac{dN}{dx} = 0$$

$$\frac{dM}{dx} = f_{QW}\, W + f_{QN}\, N + f_{QM}\, M + f_{Q} \tag{54}$$

By Equation (22), initial values for all the field functions are zero.

This problem was coded for solution on an IBM 360/50 digital computer using a forth-order Runge-Kutta integration formula [Appendix B]. A step size of 0.001 was used during the integration from x = 0 to x = 1.0, x = 1.5, and x = 2.0, and a step of 0.002 was used on the return. The execution time was 118 seconds, 144 seconds and 157 seconds for the 1000, 1500 and 2000 steps, respectively.

The general solution for W, the normal displacement of a cylindrical shell, as given in (10) is

$$W = e^{\beta x}(C_1 \cos\beta x + C_2 \sin\beta x) + e^{-\beta x}(C_3 \cos\beta x + C_4 \sin\beta x) + f(x) \tag{55}$$

where

$$\beta = \left[\frac{12(1 - \nu^3)}{4a^2h^2}\right]^{1/4}$$

$f(x)$ = particular solution

For a simply-supported cylindrical shell of finite length with a uniform internal pressure, the deflection, moment and shear equations can be obtained from Equation (55). The deflection and moment equations as given in (1) are as follows:

$$W = \frac{-p\ell^4}{64D\alpha^4}\left[1 - \frac{2\sin\alpha\sinh\alpha}{\cos 2\alpha + \cosh 2\alpha}\sin\beta x \sinh\beta x - \frac{2\cos\alpha\cosh\alpha}{\cos 2\alpha + \cosh 2\alpha}\cos\beta x \cosh\beta x\right] \tag{56}$$

$$M = \frac{-P\ell^2}{4\alpha^2}\left[\frac{\sin\alpha\sinh\alpha}{\cos 2\alpha + \cosh 2\alpha}\cosh\beta x\cos\beta x\right.$$

$$\left. - \frac{\cos\alpha\cosh\alpha}{\cos 2\alpha + \cosh 2\alpha}\sin\beta x\sinh\beta x\right] \quad (57)$$

and

$$Q = -\frac{P\ell}{2\alpha(\cos 2\alpha + \cosh 2\alpha)}\left[\sin\alpha\sinh\alpha(\sinh\beta x\cos\beta x\right.$$

$$- \cosh\beta x\sin\beta x)$$

$$\left. - \cos\alpha\cosh\alpha(\cos\beta x\sinh\beta x + \sin\beta x\cosh\beta x)\right] \quad (58)$$

where

p - uniform internal pressure

ℓ - length of the shell

D - flexural rigidity of the shell $= \dfrac{Eh^3}{12(1 - \nu^3)}$

$$\beta = \left[\frac{Eh}{4a^2D}\right]^{1/4}$$

and

$$\alpha = \frac{\beta\ell}{2}$$

All the data are the same as that previously used in the field method. Also W, M, and Q are nondimensionalized by dividing by a, Ea^2, and Ea, respectively, so the results can be directly compared with the field method results.

The expressions for W, M, and Q were evaluated on an IBM 360/50 digital computer. The expressions were evaluated at increments of 0.002 from x = 0 to x = 1.0, x = 1.5, and x = 2.0. The execution time was 10 seconds, 14 seconds, and 27 seconds respectively.

The results obtained by using the field method are compared with the exact solution in figures 4, 5, and 6 for 1000, 1500, and 2000 steps respectively, i.e. for shells of length 2a, 3a, and 4a.

Problem-2

Simply-supported thin circular cylindrical shell of finite length under a radial line load distributed around the circumference at the center section.

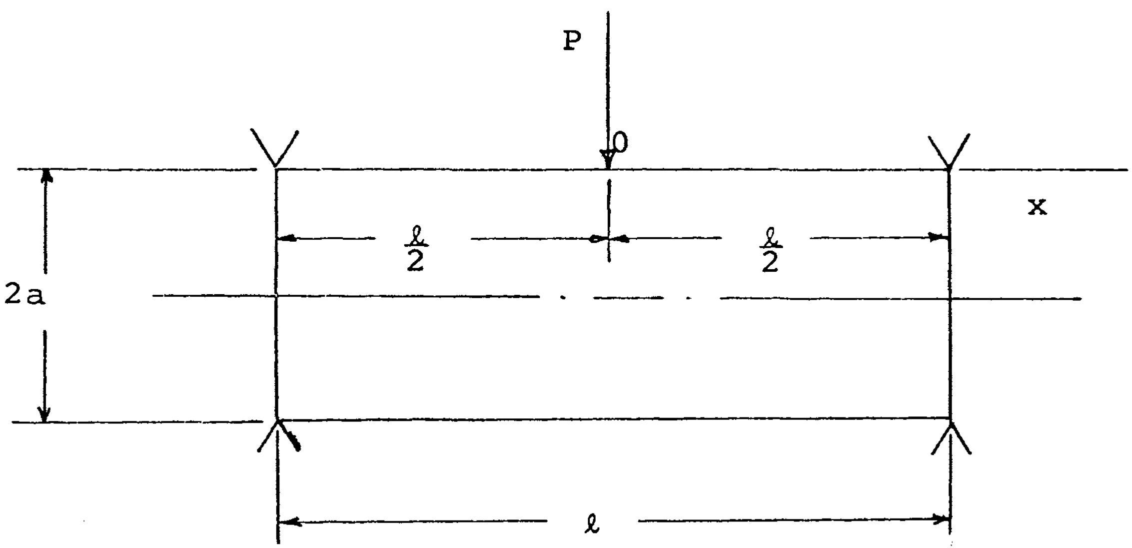


Figure 2. Cylindrical shell of finite length with both ends simply-supported and a radial line load distributed around the circumference.

The boundary conditions at the center section are:

$$(u)_{x = 0} = 0$$

$$(W')_{x = 0} = 0$$

$$(Q)_{x = 0} = - \frac{P}{2}$$

and at the supported ends

$$(W)_{x = \ell/2} = 0$$

$$(N)_{x = \ell/2} = 0$$

$$(M)_{x = \ell/2} = 0$$

All the data are the same as in problem 1. Note that P is a radial line load instead of internal pressure.

Using all the known values in the field equations (26) through (40), one obtains the same equations as (41) through (54), except Equation (52).

$$f'_Q = 0.0 - f_{QW}\, f_{W'} - f_{QM}\, f_Q \tag{52a}$$

By Equation (22) initial values for all the field functions are zero except $f_Q = Q = -P/2$.

This problem was also solved on an IBM 360/50 digital computer using a forth-order Runge-Kutta integration formula (Appendix-B). A step size of 0.001 was used during the integration from x = 0 to x = 1.0, x = 1.5, and x = 2.0, and a step of 0.002 used on the return. The execution time was 119 seconds, 146 seconds, and 165 seconds for 1000, 1500, and 2000 steps, respectively.

For a simply-supported thin cylindrical shell subjected to a line load around a circumference at the center section, the deflection, moment, and shear equations are obtained from Equation (55). The deflection equation is also given in (11).

$$\begin{aligned} W = \frac{P}{4D[2 + \cos 2\beta\ell - \cosh 2\beta\ell]} \Big[& \sin\beta x \sinh\beta x(-\sin 2\beta\ell \\ & + \sinh 2\beta\ell) \\ & + \sin\beta x \cosh\beta x\ (2 + \cos 2\beta\ell - \cosh 2\beta\ell) \\ & - \cos\beta x \sinh\beta x\ (2 + \cos 2\beta\ell - \cosh 2\beta\ell) \\ & - 2\cos\beta x \cosh\beta x\ (\sin\beta\ell \cos\beta\ell + \sinh\beta\ell \cosh\beta\ell) \Big] \end{aligned} \tag{59}$$

$$M = -\frac{P\beta}{2[2+\cos 2\beta\ell-\cosh 2\beta\ell]}\Big[\sin\beta x\,\sinh\beta x\,(\sinh 2\beta\ell$$

$$+\sin 2\beta\ell) + \cos\beta x\,\cosh\beta x\,(\sinh 2\beta\ell - \sin 2\beta\ell)$$

$$+(\cos\beta x\,\sinh\beta x + \sin\beta x\,\cosh\beta x)(2+\cos 2\beta\ell$$

$$-\cosh 2\beta\ell)\Big] \tag{60}$$

and

$$Q = -\frac{P\beta^3}{[2+\cos 2\beta\ell-\cosh 2\beta\ell]}\Big[\cos\beta x\,\sinh\beta x\,\sinh 2\beta\ell$$

$$+\sin\beta x\,\cosh\beta x\,\sin 2\beta\ell$$

$$+\cos\beta x\,\cosh\beta x\,(2+\cos 2\beta\ell-\cosh 2\beta\ell)\Big] \tag{61}$$

where

P - radial line load, ℓ, D, α and β are the same as defined in problem-1.

These expressions were also evaluated on an IBM 360/50 digital computer. They were evaluated at increments of 0.002 from x = 0 to x = 1.0, x = 1.5, and x = 2.0. The execution time was 9 seconds, 14 seconds, and 28 seconds, respectively.

The results are compared in figures 7, 8 and 9, respectively for 1000, 1500 and 2000 steps, i.e. for shells of length 2a, 3a, and 4a.

Problem-3

Thin circular cylindrical shell with both ends fixed and a uniform internal pressure.

The boundary conditions at the center section are:

$$(u)_{x=0} = 0$$

$$(W')_{x=0} = 0$$

$$(Q)_{x=0} = 0$$

and at the fixed ends

$$(W)_{x = \ell/2} = 0$$

$$(W')_{x = \ell/2} = 0$$

$$(u)_{x = \ell/2} = 0$$

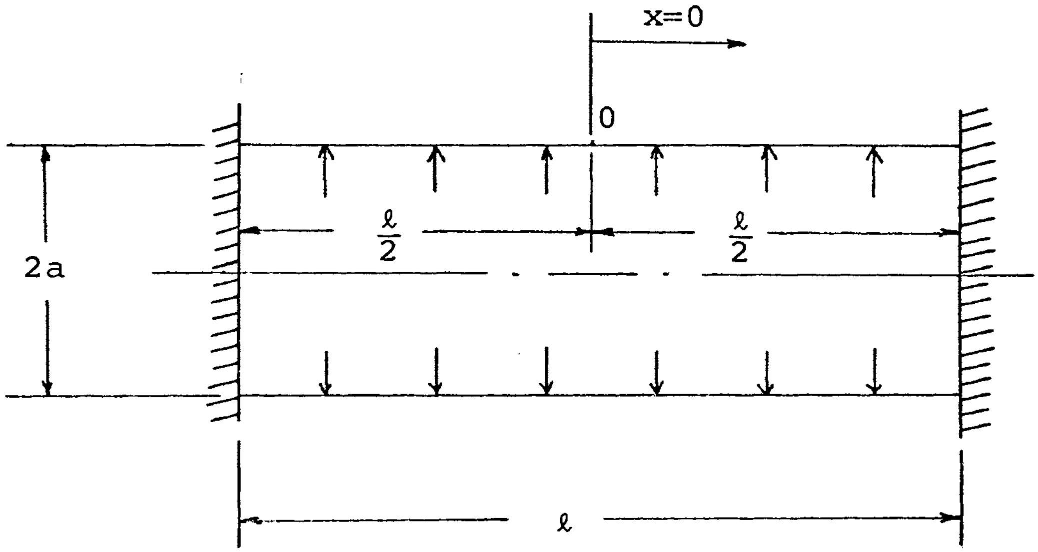


Figure 3. Cylindrical shell of finite length with both ends fixed and a uniform internal pressure.

All the data are the same as in problem-1.

Using all the known values in the field equations (26) through (40), one obtains the same equations as (41) through (52). By Equation (22), initial values for all the field functions are zero. After determining the field functions, Equations (7), (8), and (9) are used at $x = \ell/2$ to determine Q, N, and M. These equations give

$$\begin{bmatrix} 0 = f_{uN}\, N + f_{uM}\, M + f_u \\ 0 = f_{W'N}\, N + f_{W'M}\, M + f_{W'} \\ Q = f_{QN}\, N + f_{QM}\, M + f_Q \end{bmatrix}_{x = \ell/2}$$

Solving for N, M, and Q gives

$$N_{x=\ell/2} = \frac{f_u\, f_{W'M} - f_{uM}\, f_{W'}}{f_{uM}\, f_{W'N} - f_{uM}\, f_{W'M}}$$

$$M_{x=\ell/2} = \frac{f_{uN}\, f_{W'} - f_u\, f_{W'N}}{f_{uM}\, f_{W'N} - f_{uM}\, f_{W'M}}$$

$$Q_{x=\ell/2} = \frac{\begin{aligned}[t] &[f_{uN}\,(f_{QM}\, f_{W'} - f_{W'M}\, f_Q) \\ &- f_{uM}\,(f_{QN}\, f_{W'} - f_{W'N}\, f_Q) \\ &- f_u\,(f_{W'N}\, f_{QM} - f_{QN}\, f_{W'M})]\end{aligned}}{f_{uM}\, f_{W'N} - f_{uM}\, f_{W'M}}$$

This problem was also solved on an IBM 360/50 digital computer using a forth-order Runge-Kutta integration formula (Appendix-B). A step size of 0.001 was used during the integration from x = 0 to x = 1.0, x = 1.5 and x = 2.0, and a step of 0.002 was used on the return. The execution time was 129 seconds, 139 seconds and 158 seconds for 1000, 1500 and 2000 steps, respectively.

For a thin circular cylindrical shell with both ends fixed and a uniform internal pressure, the deflection, moment, and shear equation are obtained from Equation (47).

$$W = \frac{P\ell^4}{64D\alpha^4}\left[1 - \left(\frac{\sin\alpha\cosh\alpha - \cos\alpha\sinh\alpha}{\sinh\alpha\cosh\alpha + \sin\alpha\cos\alpha}\right)\sin\beta x\,\sinh\beta x - \left(\frac{\cos\alpha\sinh\alpha + \sin\alpha\cosh\alpha}{\sinh\alpha\cosh\alpha + \sin\alpha\cos\alpha}\right)\cos\beta x\,\cosh\beta x\right] \qquad (62)$$

$$M = \left[\frac{-P\ell^4\beta^2}{32\alpha^4\ (\sinh\alpha\cosh\alpha + \sin\alpha\cos\alpha)}\right]$$

$$\left[\sin\alpha\cosh\alpha\ (\cos\beta x\cosh\beta x - \sin\beta x\sinh\beta x)\right.$$

$$\left.-\cos\alpha\sinh\alpha\ (\cos\beta x\cosh\beta x + \sin\beta x\sinh\beta x)\right] \tag{63}$$

and

$$Q = \left[\frac{P\ell^4\beta^3}{16\alpha^4\ (\sinh\alpha\cosh\alpha + \sin\alpha\cos\alpha)}\right]$$

$$\left[\sin\beta x\cosh\beta x\sin\alpha\cosh\alpha\right.$$

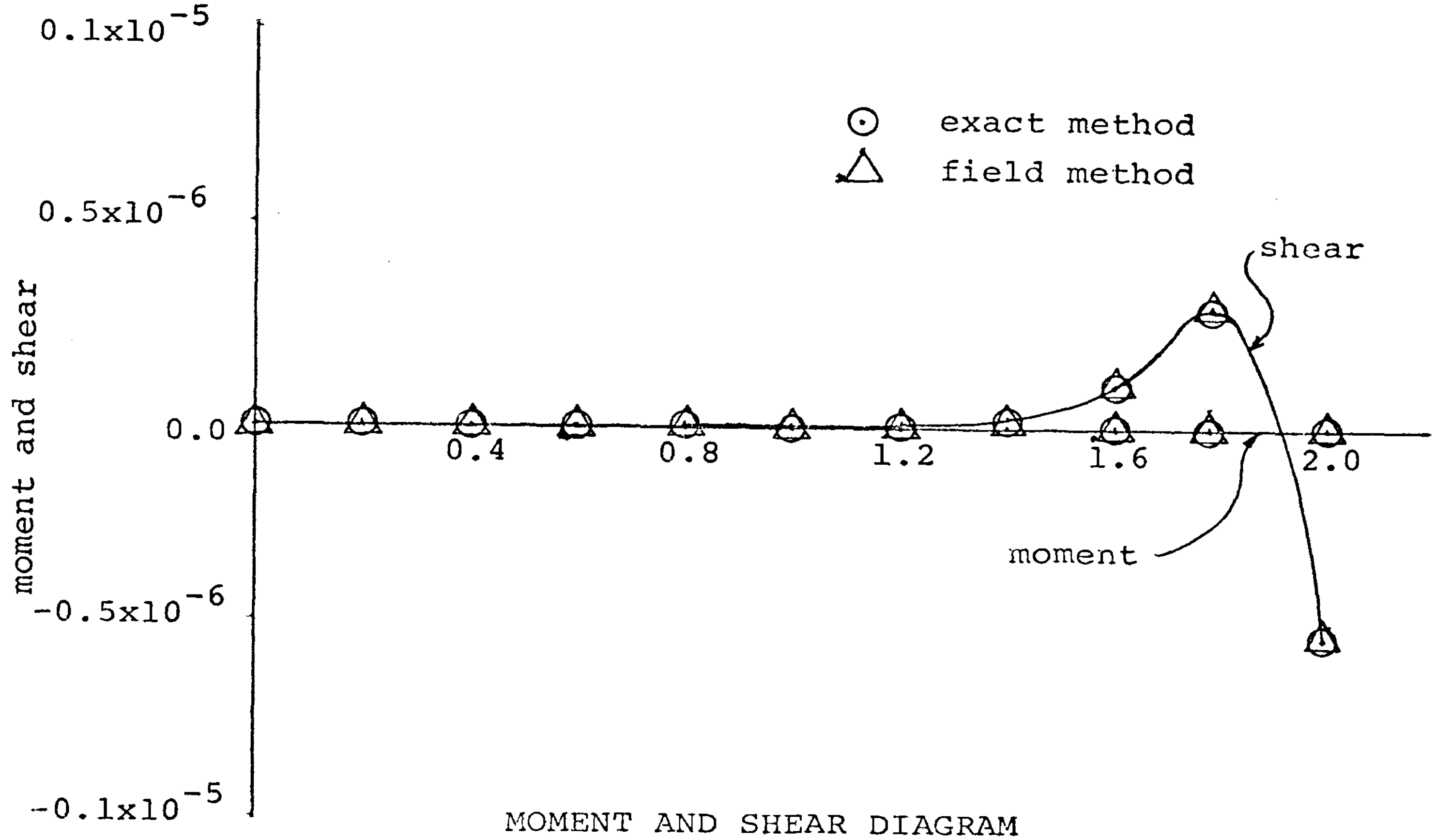
$$\left.+\cos\beta x\sinh\beta x\cos\alpha\sinh\alpha\right] \tag{64}$$

where

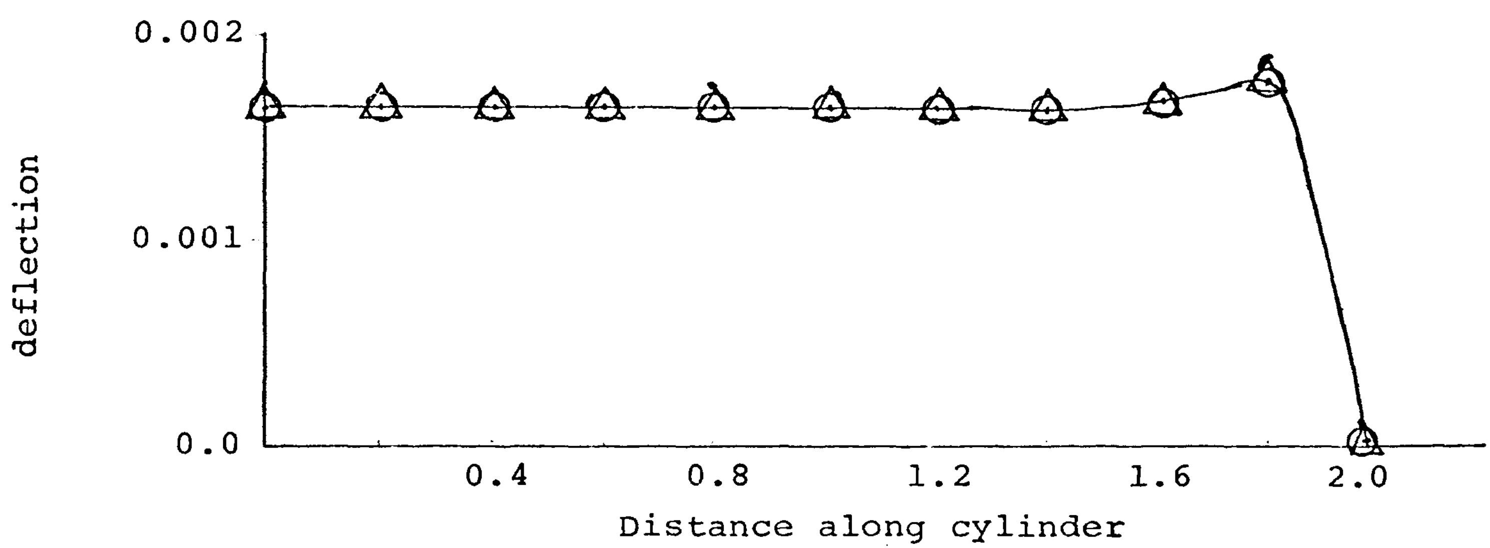
p, ℓ, D, β and α are the same as defined in problem-1.

These expressions were evaluated on an IBM 360/50 digital computer. They were evaluated at increments of 0.002 from $x = 0$ to $x = 1.0$, $x = 1.5$, and $x = 2.0$. The execution time was 9 seconds, 14 seconds, and 29 seconds, respectively.

The results are compared in figures 10, 11 and 12, respectively for 1000, 1500, and 2000 steps, i.e. for shells of length 2a, 3a, and 4a.



MOMENT AND SHEAR DIAGRAM



DEFLECTION DIAGRAM

Figure 4. Comparison of the field method with the exact method curve, for 2a long cylinder (problem-1).

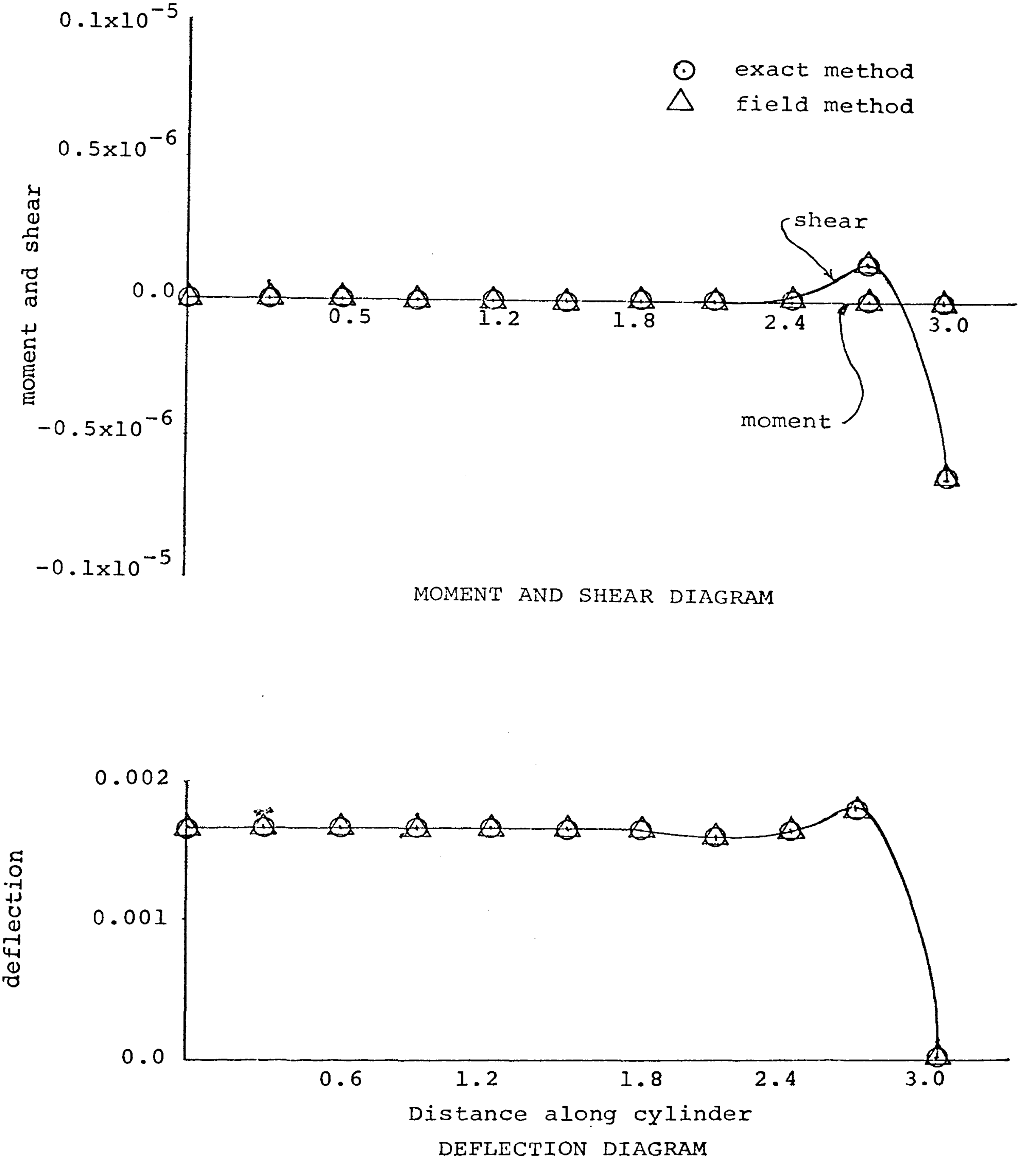


Figure 5. Comparison of the field method with the exact method curve, for 3a long cylinder (problem-1).

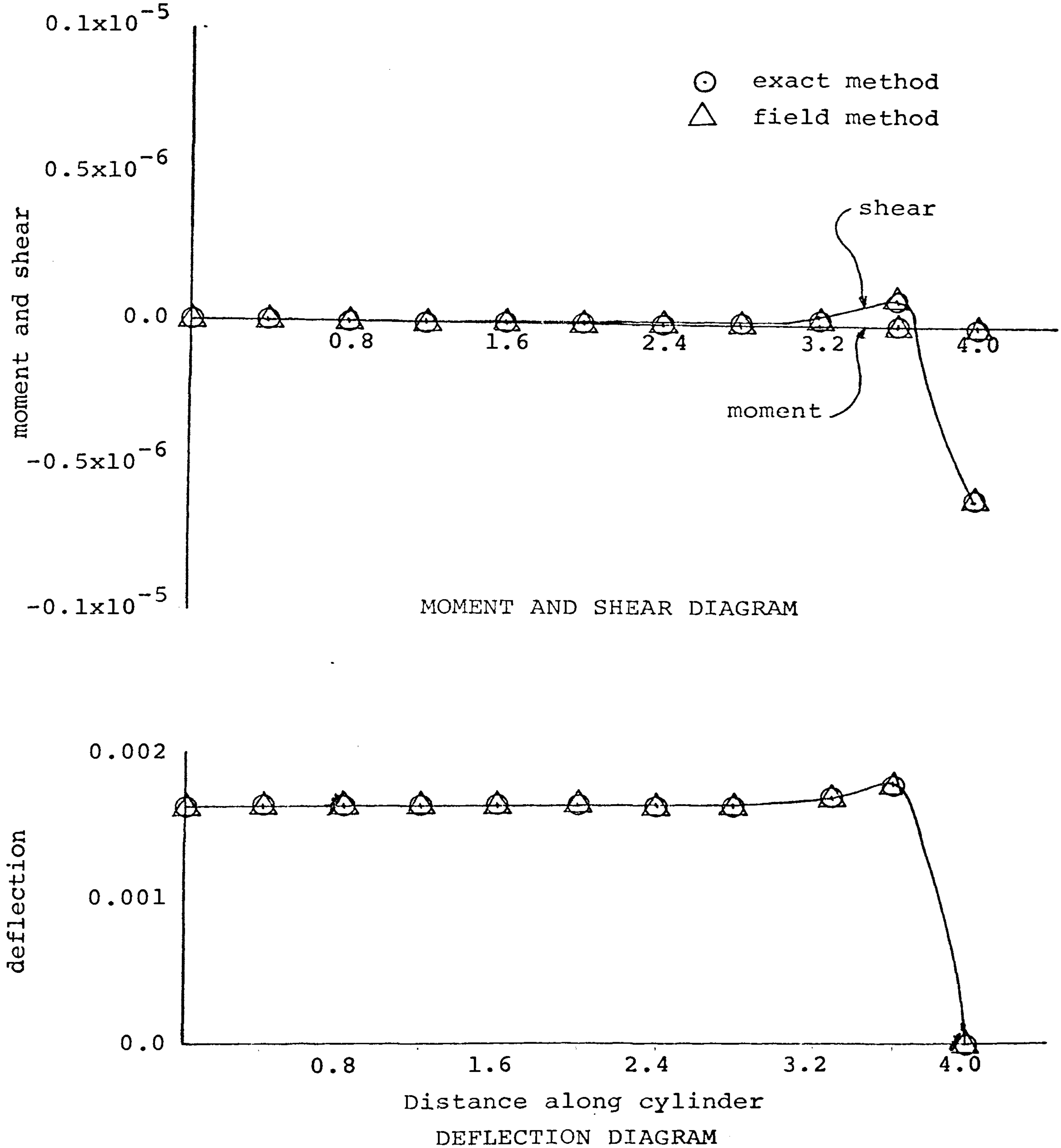
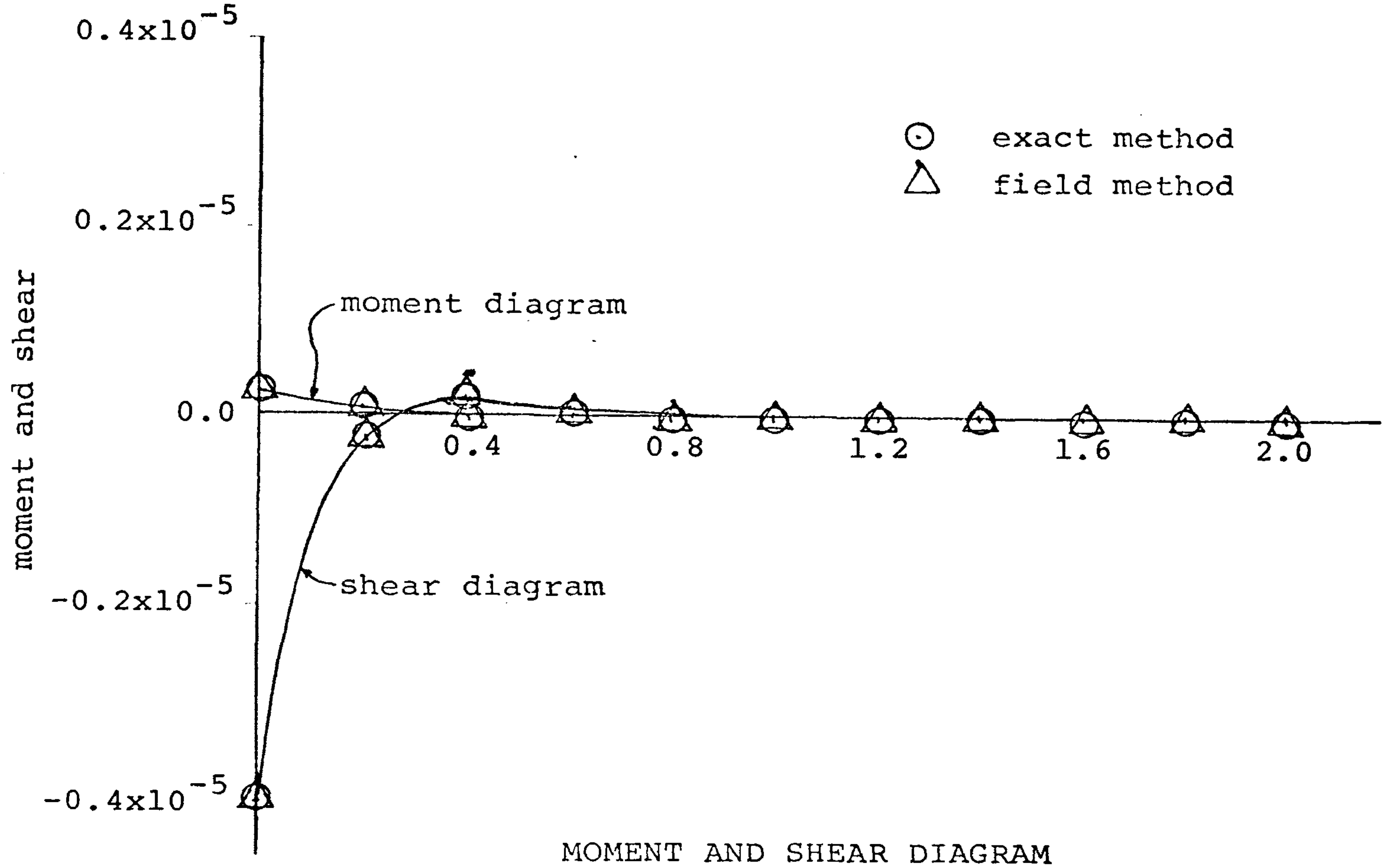
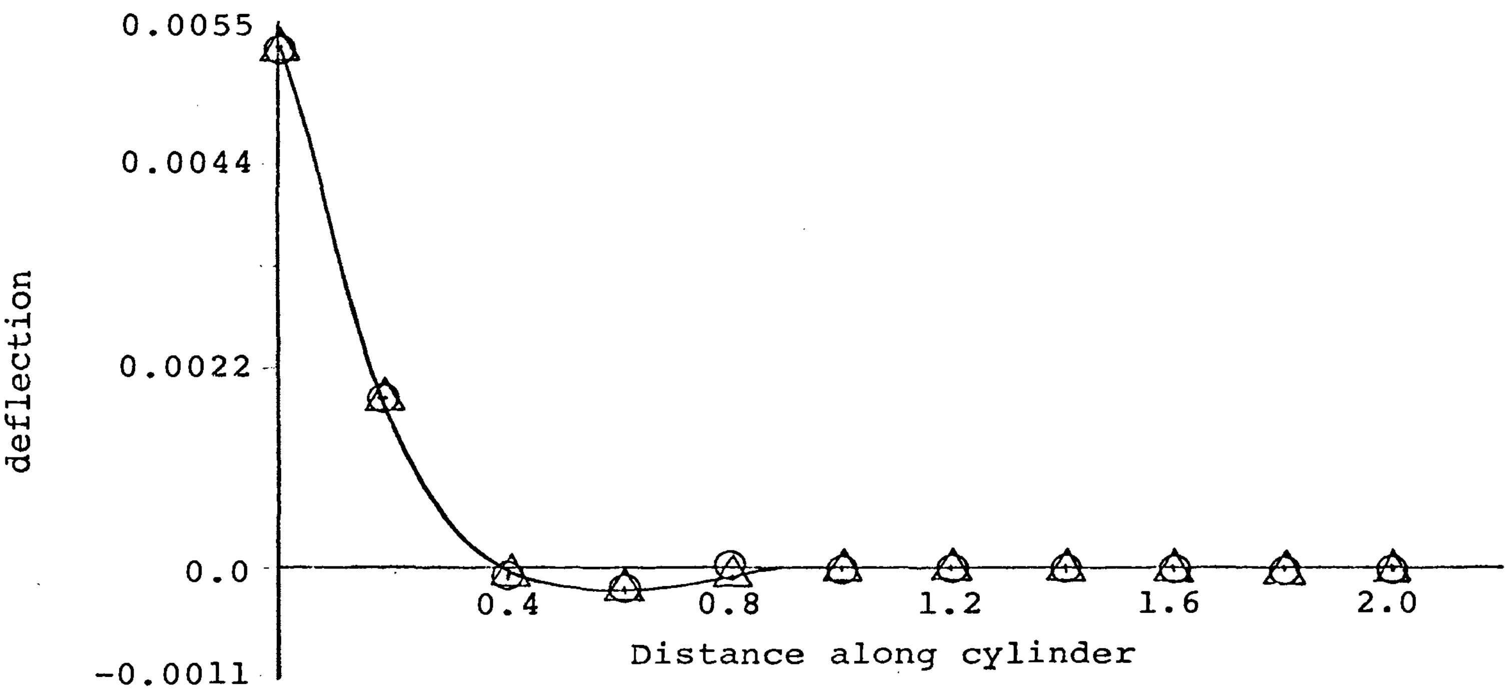


Figure 6. Comparison of the field method with the exact method curve, for 4a long cylinder (problem-1).



MOMENT AND SHEAR DIAGRAM



DEFLECTION DIAGRAM

Figure 7. Comparison of the field method with the exact method curve, for 2a long cylinder (problem-2).

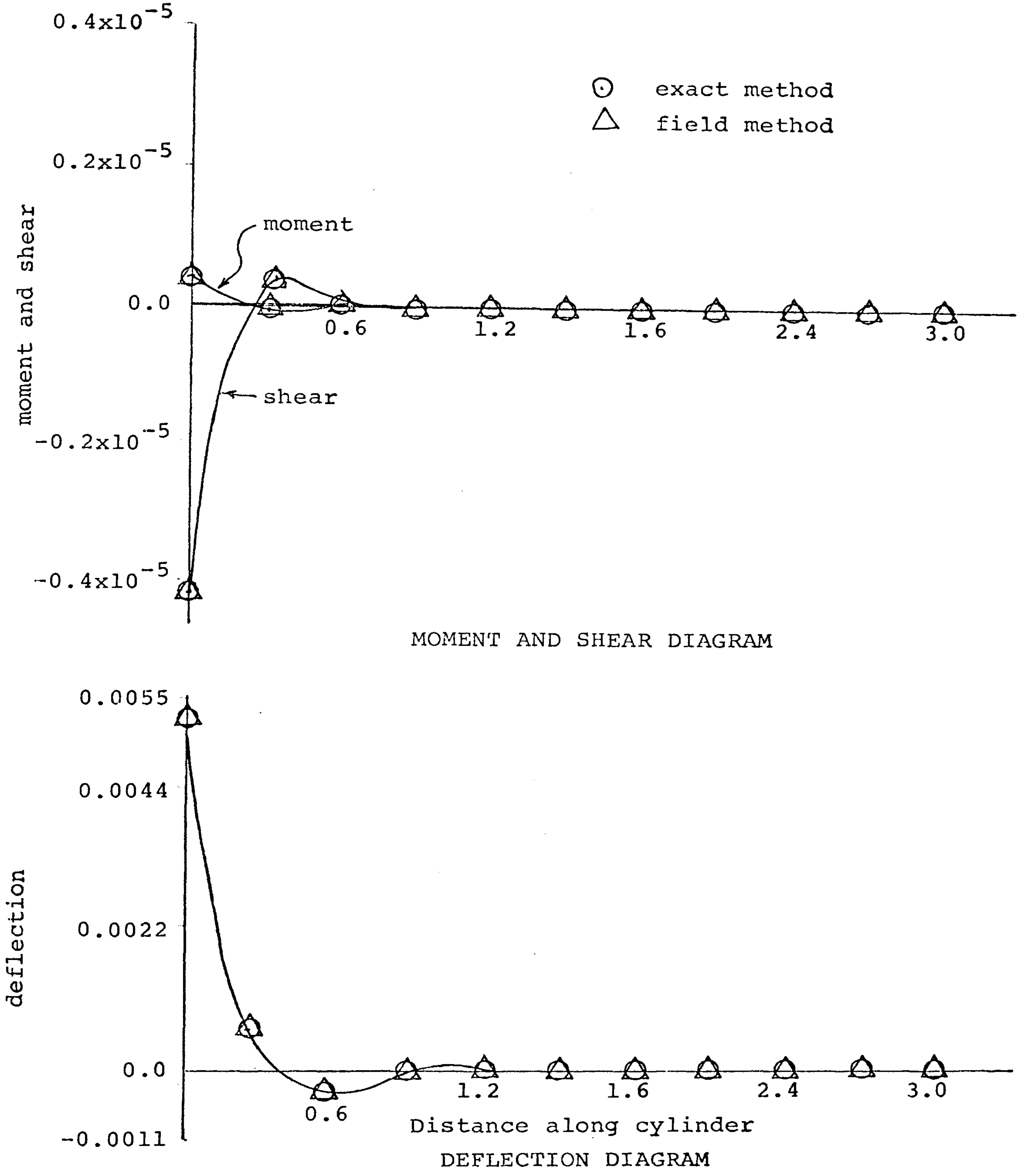


Figure 8. Comparison of the field method with the exact method curve, for 3a long cylinder (problem-2).

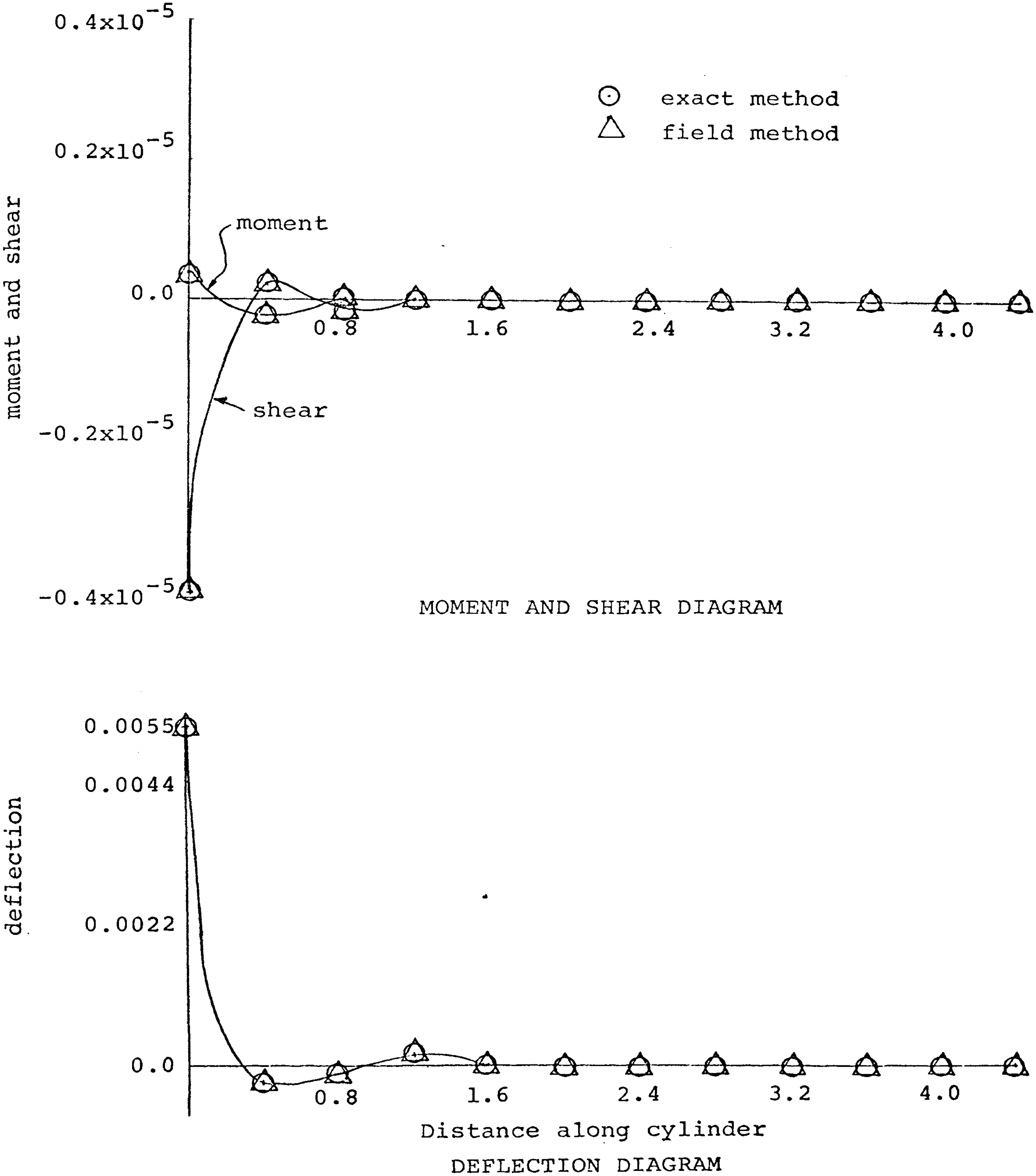


Figure 9. Comparison of the field method with the exact method curve, for 4a long cylinder (problem-2).

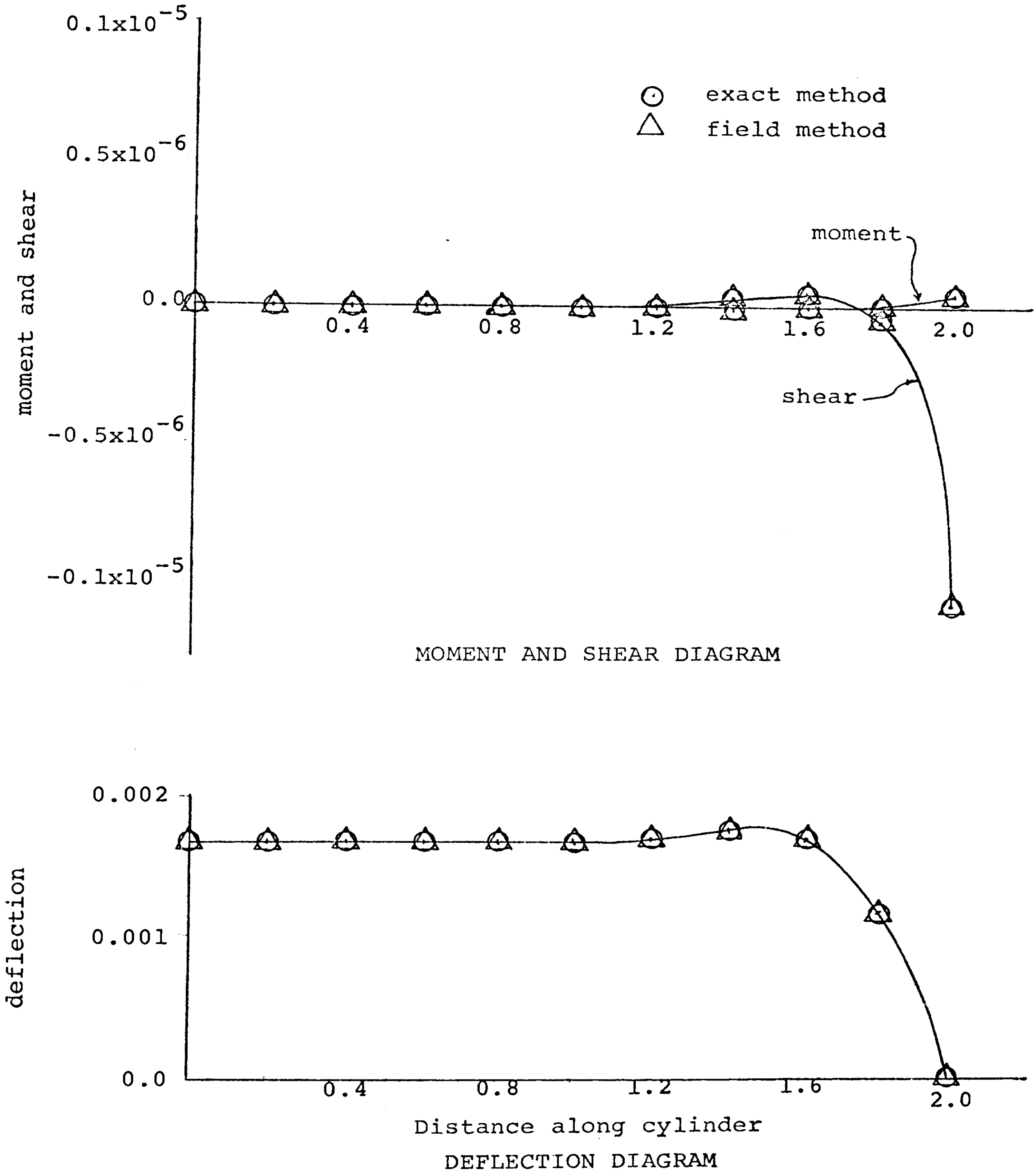


Figure 10. Comparison of the field method with the exact method curve, for 2a long cylinder (problem-3).

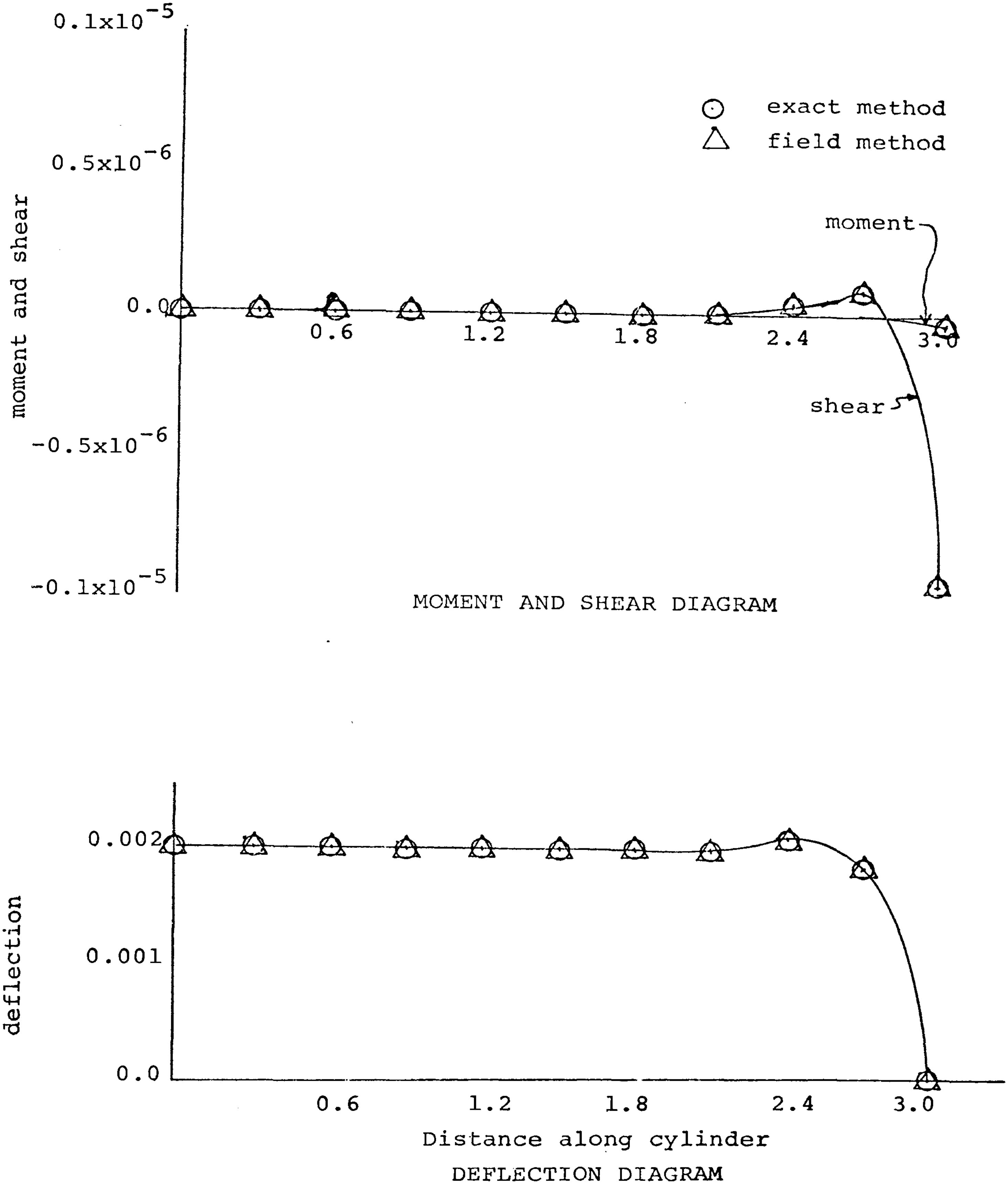
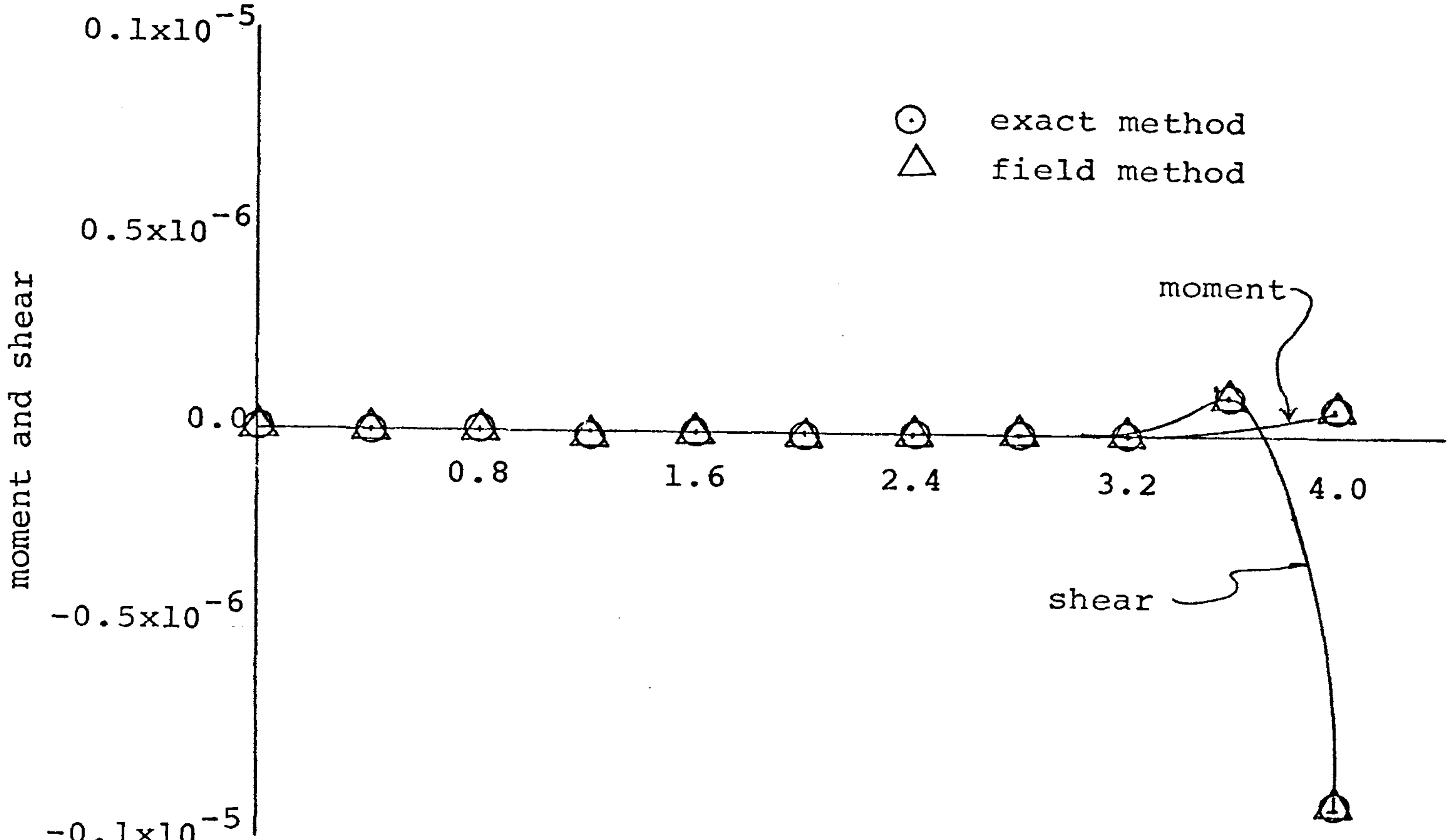
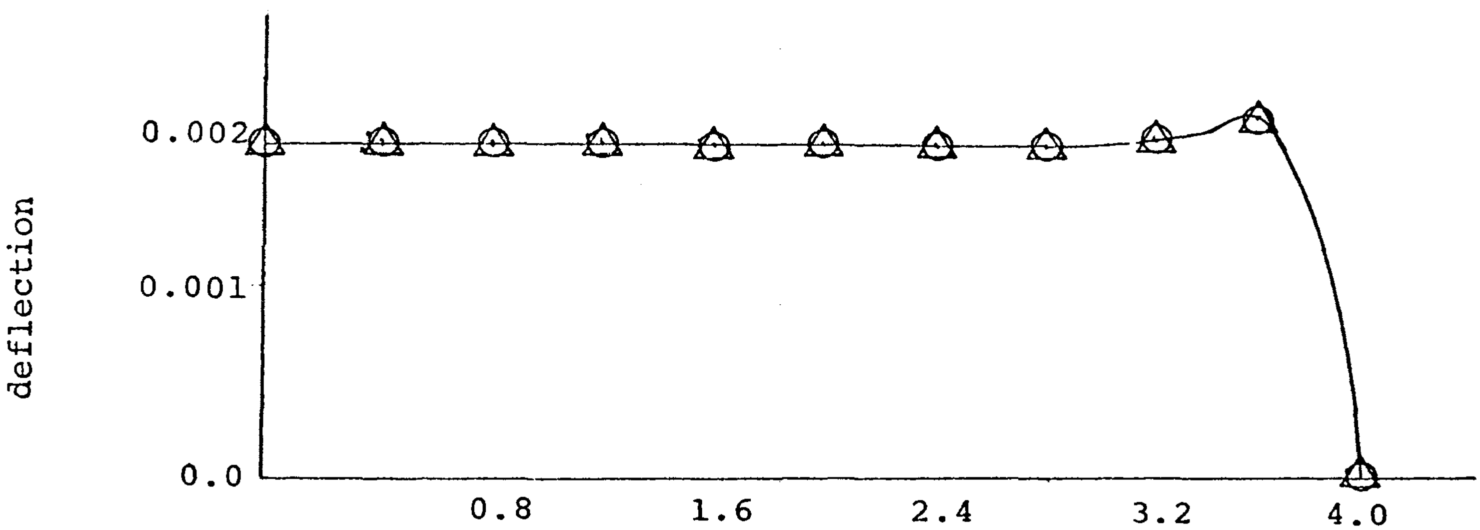


Figure 11. Comparison of the field method with the exact method curve, for 3a long cylinder (problem-3).



MOMENT AND SHEAR DIAGRAM



DEFLECTION DIAGRAM

Figure 12. Comparison of the field method with the exact method curve, for 4a long cylinder (problem-3).

V. CONCLUSION

The results obtained by using the field method for the displacements of a long cylinder with simply supported, and with fixed ends, are in very good agreement at all points throughout the shell with the exact solution. The results for the moments and shear forces differ slightly from the exact solution. However, this difference occurs only when they are very small (i.e. 1×10^{-9} and less). In fact the moments and shear forces are essentially zero in these regions. Thus the results obtained by using the field method to solve the shell equations for a cylinder are very good.

Since the solutions to the shell equations behave similarly for any type of shell geometry, it appears that the field method might provide a very powerful means of solving static and dynamic shell problems. Additional examples should be investigated using geometries and loadings for which solution are known.

BIBLIOGRAPHY

1. MILLER, R. E., "Use of the Field Method for Numerical Integration of Two-point Boundary-Value Problems". AIAA Journal, Vol. 5, No. 4, April 1967, p. 811-813.

2. NAGHDI, P. M., and C. N. DESILVA, "Deformation of Elastic Ellipsoidal Shells of Revolution". Proceedings of the Second U.S. National Congress of Applied Mechanics, 1954, p. 333-343.

3. WILSON, P. E., and E. S. EDWARD, "Numerical Analysis of Large Axisymmetric Deformations of Thin Spherical Shells". AIAA Journal, Vol. 3, No. 9, September 1965, p. 1717-1725.

4. GALLETLY, G. D., W. T. KYNER, and C. E. MOLLER, "Numerical Methods and the Bending of Ellipsoidal Shells". Journal of Society of Industrial and Applied Mathematics, Vol. 9, 1961, p. 489-513.

5. RADKOWSKI, P. P., R. M. DAVIS, and M. R. BOLDUC, "Numerical Analysis of Equations of Thin Shells of Revolution". American Rocket Society Journal, Vol. 32, 1962, p. 36-41.

6. SEPETOSKI, W. K., C. E. PEARSON, I. W. DINGWELL, and A. W. ADKINS, "A Digital Computer Program for the General Axially Symmetric Thin-Shell Problem". Journal of Applied Mechanics, Vol. 29, Trans. ASME, Vol. 84, Series E, 1962, p. 655-661.

7. KALNINS, A., "Analysis of Shells of Revolution Subjected to Symmetrical and Nonsymmetrical loads". Journal of Applied Mechanics, September 1964, p. 467-476.

8. KEITH, H. D., "Minimum Weight Design of Axially Symmetric Shells of Revolution". Ph.D. Thesis, University of Illinois.

9. LANGHAAR, H. L., "Foundations of Practical Shell Analysis". Department of Theoretical and Applied Mechanics, University of Illinois Press. Revised Edition, June 1964.

10. TIMOSHENKO, S., and WOINOWSKY-KRIEGER, S., "Theory of Plates and Shells". New York, McGraw-Hill, 1959.

11. MECK, H. R., "Bending of a Thin Cylindrical Shell Subjected to a Line Load Around a Circumference". Journal of Applied Mechanics Volume 28, Trans. ASME, Series E, September 1961, p. 427-433.

APPENDICES

APPENDIX-A

A LINEAR BENDING THEORY FOR SHELLS OF REVOLUTION

In this appendix a summary of the derivation of the linear ordinary differential equations for the displacements and stress-resultants in an elastic shell segment subjected to an axially symmetric loading is presented. In this derivation it has been assumed that the shell is made of a linearly elastic, homogeneous, isotropic material and undergoes small strains.

For a shell of revolution it is convenient to use the longitude θ of a meridian and the arc length s measured from the vertex as shell coordinates (see Figure 13).

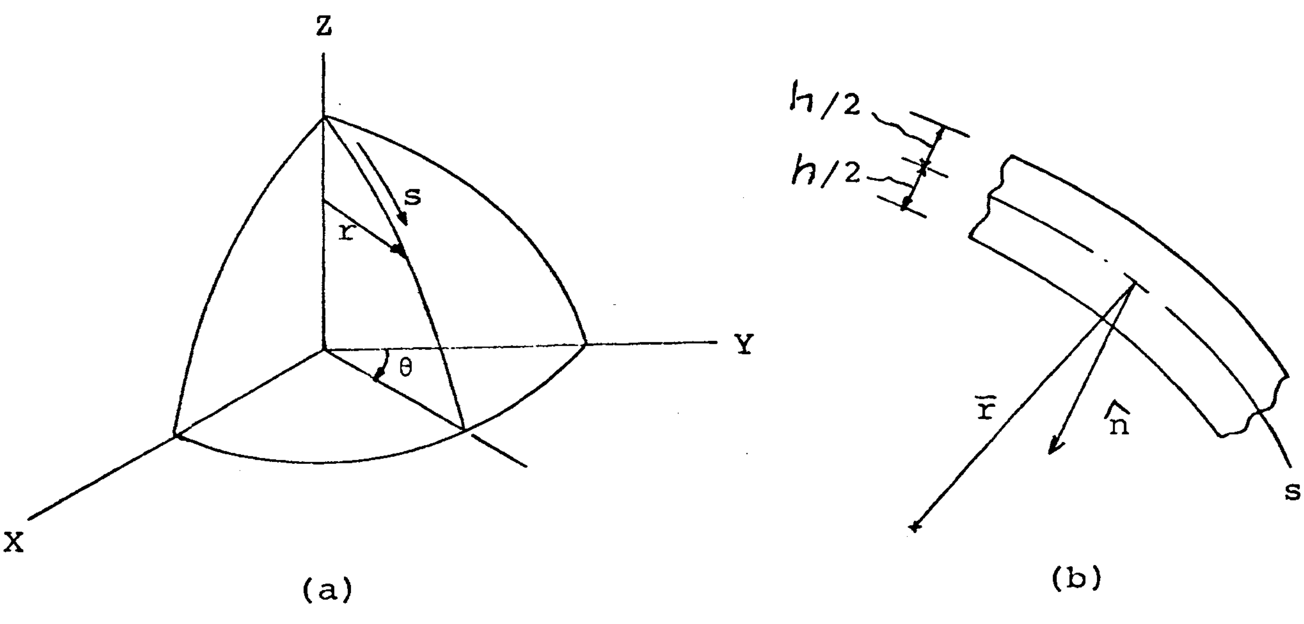


Figure 13. Shell Coordinates.

If the radius of a cross-sectional circle with coordinate s is denoted by r, the equations of a meridian are

$$r = r(s), \qquad z = z(s)$$

The general equilibrium equations for orthogonal shell coordinates are (9)

$$\frac{\delta}{\delta s}[BN_s] + \frac{\delta}{\delta\theta}[AN_{\theta s}] + A_\theta N_{s\theta} - B_s N_\theta$$

$$\frac{-Be}{A} Q_s + ABP_s = 0 \tag{1a}$$

$$\frac{\delta}{\delta s}[BN_{s\theta}] + \frac{\delta}{\delta\theta}[AN_\theta] - A_\theta N_s + B_s N_{\theta s}$$

$$-fQ_s - \frac{Ag}{B} Q_\theta + ABP_\theta = 0 \tag{2a}$$

$$\frac{\delta}{\delta s}[BQ_s] + \frac{\delta}{\delta\theta}[AQ_\theta] + \frac{Be}{A} N_s + \frac{Ag}{B} N_\theta$$

$$+ (N_{s\theta} + N_{\theta s}) f + ABP_z = 0 \tag{3a}$$

$$\frac{\delta}{\delta s}[BM_s] + \frac{\delta}{\delta\theta}[AM_{\theta s}] + A_\theta M_{s\theta}$$

$$- B_s M_\theta + ABL_\theta = ABQ_s \tag{4a}$$

$$\frac{\delta}{\delta s}[BM_{s\theta}] + \frac{\delta}{\delta\theta}[AM_\theta] - A_\theta M_s$$

$$+ B_s M_{\theta s} - ABL_s = ABQ_\theta \tag{5a}$$

$$N_{s\theta} - N_{\theta s} - \frac{Q}{A^2} M_{s\theta} + \frac{f}{AB}(M_s - M_\theta) + \frac{g}{B^2} M_{\theta s} = 0 \tag{6a}$$

The terms M_s, M_θ, $M_{s\theta}$, $M_{\theta s}$, N_s, N_θ, $N_{s\theta}$, $N_{\theta s}$, Q_s, Q_θ are defined in (9). As indicated in Figure 25 of (9) N_s and N_θ are the resultant tensile forces per unit length of a coordinate line, Q_s is the transverse shear, M_s and M_θ are the bending moments, and P is the load per unit area of the coordinate (middle) surface.

Since the coordinate lines (meridians and cross-sectional circles) are lines of principal curvature,

$$A = 1 \qquad e = 1/r_1$$
$$B = r \qquad g = r^2/r_2$$
$$F = 0 \qquad f = 0$$

(where r_1 and r_2 are the principal raddi of curvature of the middle surface of the shell.)

For axially symmetrical loading, the quantities $M_{s\theta}$, $M_{\theta s}$, $N_{s\theta}$, $N_{\theta s}$, Q_θ, L_s, P_θ, and $\frac{\delta}{\delta\theta}$ are zero. With these simplifications equations (1a) through (6a) reduce to

$$\frac{d}{ds}[rN_s] - r'N_\theta - \frac{r}{r_1}Q_s + rP_s = 0 \tag{7a}$$

$$\frac{d}{ds}[rQ_s] + \frac{r}{r_1}N_s + \frac{r}{r_2}N_\theta + rP_z = 0 \tag{8a}$$

$$\frac{d}{ds}[rM_s] - r'M_\theta - rQ_s = 0 \tag{9a}$$

Derivatives of r(s) and z(s) with respect to s are denoted by r' and z', respectively.

Neglecting thermal effects, the Stress-Strain relations are (9):

$$N_s = \frac{Eh}{1-\nu^2}[\varepsilon_s + \nu\varepsilon_\theta]$$

$$N_\theta = \frac{Eh}{1-\nu^2}[\varepsilon_\theta + \nu\varepsilon_s]$$

$$M_s = \frac{Eh^3}{12(1-\nu^2)}[K_s + \frac{\nu}{r^2}K_\theta]$$

$$M_\theta = \frac{Eh^3}{12(1-\nu^2)}[\frac{K\theta}{r^2} + \nu K_s]$$

The terms ε_s and ε_θ are the strain components of the middle surface of the shell, and K_s and K_θ are related to the changes of curvature of the middle surface of the shell caused by bending.

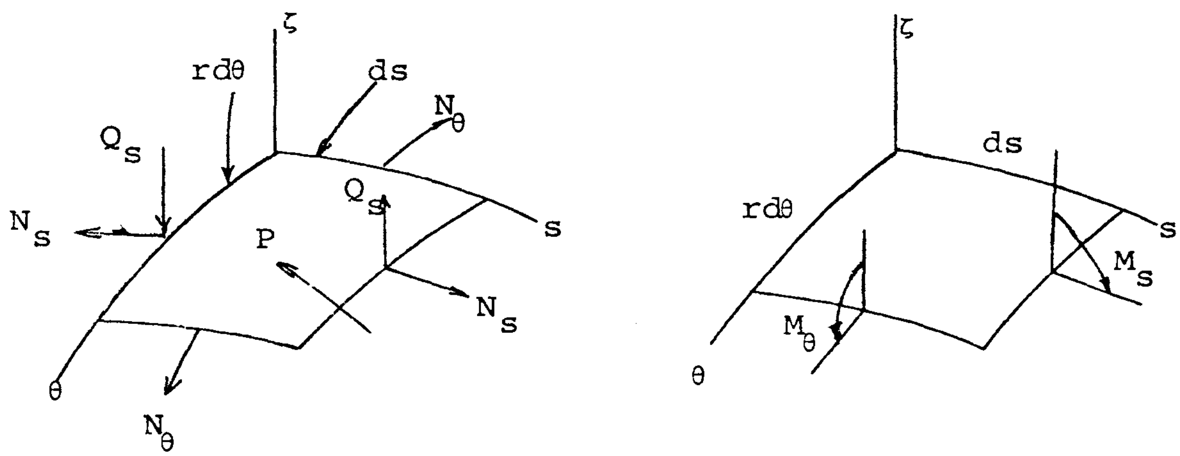


Figure 14. Stress Resultants.

The linear strain-displacement relations are:

$$\varepsilon_s = u' - W/r_1$$

$$\varepsilon_\theta = r'u/r - W/r_2$$

$$K_s = W''$$

$$K_\theta = rr'W'$$

where primes represent derivative with respect to s. The displacement component u is in the s-direction, and W is in the n-direction.

The equilibrium equations, stress-strain relations, and the strain-displacment relations can be combined to give the following first-order system of differential equations for u, W, W', N_s, M_s, and Q_s:

$$\frac{du}{ds} = -\frac{\nu r'u}{r} + \left[\frac{1}{r_1} + \frac{\nu}{r_2}\right] W + \left[\frac{1-\nu^2}{h}\right] N$$

$$\frac{dW}{ds} = W'$$

$$\frac{dW'}{ds} = -\frac{\nu r'}{r} W' - \frac{12(1-\nu^2)}{h^3} M$$

$$\frac{dN}{ds} = \frac{hr'^2}{r^2} u - \frac{hr'}{rr_2} W - \frac{r'}{r} [1-\nu] N + \frac{Q}{r_1} - P_s$$

$$\frac{dM}{ds} = -\frac{h^3 r'^2}{12r^2} W' - \frac{r'}{r} [1-\nu] M + Q$$

$$\frac{dQ}{ds} = -\frac{hr'}{rr_2} u + \frac{h}{r_2^2} W - \left[\frac{1}{r_1} + \frac{\nu}{r_2}\right] N - \frac{r'}{r} Q - P_z$$

These equations have been nondimensionalized such that

$u \rightarrow u/r_f$	$r \rightarrow r/r_f$	$P_s \rightarrow P_s/E$
$W \rightarrow W/r_f$	$r' \rightarrow r'/r_f$	$P_z \rightarrow P_z/E$
$N \rightarrow N_s/r_f E$	$r_1 \rightarrow r_1/r_f$	$Q \rightarrow Q_s/r_f E$
$M \rightarrow M_s/r_f^2 E$	$r_2 \rightarrow r_2/r_f$	$h \rightarrow h/r_f$

APPENDIX-B

COMPUTER PROGRAM FOR INTEGRATING SHELL EQUATIONS BY THE FIELD METHOD

The following program is written in the Basic Operating System Fortran IV language for the IBM System/360. A forth-order Runge-Kutta integration formula is used. The program was executed on the IBM 360/50 at the University of Missouri at Rolla.

The following input data is required:

(1) The integration step size, STEP

(2) Number of steps, i.e., Length of the shell.

(3) Shell Geometry; $\frac{1}{r_1}$ = AG, $\frac{1}{r_2}$ = BG, r = CG, and r' = DG

(4) Shell thickness, H

(5) Internal Pressure P/E = POE

(6) Poisson's ratio μ = GNU, and

(7) Initial values of the field functions.

The program integrates the shell equations for three problems with three sets of boundary conditions at the center section and at the end.

For problem-1, the boundary conditions at the center section are:

$$(u)_{x = 0} = 0$$

$$(W')_{x = 0} = 0$$

$$(Q)_{x = 0} = 0$$

and at the supported ends

$$(W)_{x = \ell/2} = 0$$

$$(N)_{x = \ell/2} = 0$$

$$(M)_{x = \ell/2} = 0$$

For problem-2, the boundary conditions at the center section are:

$$(u)_{x = 0} = 0$$

$$(W')_{x = 0} = 0$$

$$(Q)_{x = 0} = -P/2$$

and at the supported ends

$$(W)_{x = \ell/2} = 0$$

$$(N)_{x = \ell/2} = 0$$

$$(M)_{x = \ell/2} = 0$$

For problem-3, the boundary conditions at the center section are:

$$(u)_{x = 0} = 0$$

$$(W')_{x = 0} = 0$$

$$(Q)_{x = 0} = 0$$

and at the fixed ends

$$(W)_{x = \ell/2} = 0$$

$$(W')_{x = \ell/2} = 0$$

$$(u)_{x = \ell/2} = 0$$

The output data are W, N, M, u, W', and Q at each point.

PROGRAM FOR INTEGRATING SHELL EQUATIONS BY FIELD METHOD

```
C       NUM=NUMBER OF STEPS
C       NE=NUMBER OF EQUATIONS
C       FOLLOWING TWELVE ARE THE FIELD FUNCTIONS, WHERE D
        REPRESENTS DERIVATIVE.
C       DFUW=XF(1)
C       DFUN=XF(2)
C       DFUM=XF(3)
C       DFU=XF(4)
C       DFW1W=XF(5)
C       DFW1N=XF(6)
C       DFW1M=XF(7)
C       DFW1=XF(8)
C       DFQW=XF(9)
C       DFQN=XF(10)
C       DFQM=XF(11)
C       DFQ=XF(12)
C       DW=ZF(13)
C       DN=ZF(14)
C       DM=ZF(15)
C       XF(1),XF(2)...XF(12), ARE THE DERIVATIVES OF THE FIELD
        FUNCTIONS, AND ZF(13)...ZF(15), ARE THE DERIVATIVES FOR
        W, N, M, BUT FOR SIMPLE (WITHOUT DERIVATIVE) FUNCTIONS,
        THEY ARE X(1),X(2)...X(12), Z(13)...Z(14).
        DIMENSION X(12),XK1(12),XK2(12),XK3(12),XK4(12),XF(12),
        XK(12),XU(2002),XQ(2002),Z(15),ZK1(15),ZK2(15),ZK3(15),
        ZK4(15),ZF(15),ZK(15)
        COMMON YY(12,2002)
*       READ (1,400) STEP, NUM, NE, (YY(I,1),I=1,12),
        (YY(I,NUM),I=13,15)
*       WRITE (3,400) STEP, NUM, NE,(YY(I,1),I=1,12),
        (YY(I,NUM), I=13,15)
    2   DO  10  J=2, NUM
        DO  20  I=1, NE
   20   X(I)=YY(I,J-1)
        CALL DERV (X,XF)
        DO  30  I=1, NE
   30   XK1(I)=STEP* XF(I)
```

*For problem three these statements are different.

READ(1,400) STEP, NUM, NE, (YY(I,1), I=1,12),

WRITE (3,400) STEP, NUM, NE, (YY(I,1), I=1,12),

```
      DO  40  I=1, NE
   40 X(I)=YY(I,J-1)+XK1(I)/2.
      CALL DERV (X,XF)
      DO  50  I=1, NE
   50 XK2(I)=STEP* XF(I)
      DO  60  I=1, NE
   60 X(I)=YY(I,J-1)+XK2(I)/2.
      CALL DERV (X,XF)
      DO  70  I=1, NE
   70 XK3(I)=STEP* XF(I)
      DO  80  I=1, NE
   80 X(I)=YY(I,J-1)+XK3(I)
      CALL DERV (X,XF)
      DO  90  I=1, NE
   90 XK4(I)=STEP* XF(I)
      DO  100  I=1, NE
  100 XK4(I)=(XK2(I)+2.*(XK2(I)+XK3(I))+XK4(I))/6.
      DO  200  I=1, NE
  200 YY(I,J)=YY(I, J-1)+XK(I)
   10 CONTINUE
*     D=YY(3,NUM)*YY(6,NUM)-YY(2,NUM)*YY(7,NUM)
*     YY(13,NUM)=0.0
*     YY(14,NUM)=(YY(4,NUM)*YY(7,NUM)-YY(3,NUM)*YY(8,NUM))/D
*     YY(15,NUM)=(YY(2,NUM)*YY(8,NUM)-YY(4,NUM)*YY(6,NUM))/D
      L=NUM
      DO  15  J=1,L,2
      K=(NUM+1)-J
      DO  25  I=13,15
   25 Z(I)=YY(I,K)
      CALL GRANT (Z,ZF,K)
      DO  35  I=13,15
   35 ZK1(I)=(-2.*STEP)*ZF(I)
      DO  45  I=13,15
   45 Z(I)=YY(I,K)+ZK1(I)/2.
      CALL GRANT (Z,ZF,K-1)
      DO  55  I=13,15
   55 ZK2(I)=(-2.*STEP)* ZF(I)
      DO  65  I=13,15
   65 Z(I)=YY(I,K)+ZK2(I)/2.
      CALL GRANT (Z,ZF,K-1)
      DO  75  I=13,15
   75 ZK3(I)=(-2.*STEP)*ZF(I)
      DO  85  I=13,15
   85 Z(I)=YY(I,K)+ZK3(I)
      CALL GRANT (Z,ZF, K-2)
      DO  95  I=13,15
   95 ZK4(I)=(-2.*STEP)* ZF(I)
      DO  150  I=13,15
  150 ZK(I)=(ZK1(I)+2.*(ZK2(I)+ZK3(I))+ZK4(I))/6.
```

---

*These statements are only for problem three.

```
      DO  250  I=13,15
  250 YY(I,K-2)=YY(I,K)+ZK(I)
      WRITE (3,600)(YY(I,K-2), I=13,15)
  120 XU(K-2)=(YY(1,K-2)*YY(13,K-2))
      +(YY(2,K-2)*YY(14,K-2))+(YY(3,K-2)*
      YY(15,K-2))+YY(4,K-2)
      WRITE (3,700) XU(K-2)
  130 XQ(K-2)  (YY(9,K-2)*YY(13,K-2))+
      (YY(10,K-2)*YY(14,K-2))+(YY(11,K-2)*
      YY(15,K-2))+YY(12,K-2)
      WRITE (3,700)XQ(K-2)
   15 CONTINUE
      STOP
  400 FORMAT (F8.4,2I8,/12F6.2,/2F6.2)
  600 FORMAT (//18X,/3E15.7)
  700 FORMAT (18X,/3E15.7)
      END
      SUBROUTINE DERV (X,XF)
      DIMENSION XF(12), X(12)
      GNU=0.25
      CG=1.0
      H= 0.01
*     POE=500/30000000

      XF(1)=GNU/CG-(X(1)*X(5))-(X(3)*X(9))

      XF(2)=(1-GNU*GNU)/H-(X(1)*X(6))-(X(3)*X(10))

      XF(3)=-((X(1)*X(7))+(X(3)*X(11)))

      XF(4)=-(X(1)*X(8))-(X(3)*X(12))

      XF(5)=-(X(5)*X(5))-(X(7)*X(9))

      XF(6)=-(X(6)*X(5))-(X(7)*X(10))

      XF(7)=-12.(1-GNU*GNU)/H**3-(X(5)*X(7))-(X(7)*X(11))

      XF(8)=-(X(5)*X(8))-(X(7)*X(12))

      XF(9)=H/CG*CG-(X(9)*X(5)-(X(11)*X(9))

      XF(10)=-GNU/CG-(X(9)*X(6))-(X(11)*X(10))

      XF(11)=-(X(11)*X(11))-(X(9)*X(7))

      XF(12)=-POE-(X(9)*X(8))-(X(11)*X(12))
      RETURN
      END
      SUBROUTINE GRANT (Z,ZF,LOC)
      DIMENSION ZF(15),Z(15)
```

*POE for second problem is zero.

```
      COMMON YY(12,2002)
      ZF(13)=(YY(5,LOC)*Z(13))+(YY(6,LOC)*Z(14))+(YY(7,LOC)*
             Z(15))+YY(8,LOC)
      ZF(14)=0.0
      ZF(15)=(YY(9,LOC)*Z(13)+(YY(10,LOC)*Z(14))+(YY(11,LOC)
             XZ(15))+YY(12,LOC)
      RETURN
      END
/DATA
/END
```

TABLE-1

A Comparison Between Exact and Field Solutions for 2a Long Cylinder (Problem-1)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.1667E-02	0.1667E-02	0.1015E-12	0.1321E-12	0.0000E 79	0.0000E 79
100	0.2000E 00	0.1667E-02	0.1667E-02	-0.2279E-12	-0.1514E-12	-0.7189E-11	-0.8470E-11
200	0.4000E 00	0.1667E-02	0.1667E-02	-0.1182E-11	-0.1036E-11	-0.6949E-11	-0.6764E-11
300	0.6000E 00	0.1667E-02	0.1667E-02	0.1232E-11	0.1294E-11	0.7994E-10	0.7770E-10
400	0.8000E 00	0.1667E-02	0.1667E-02	0.1861E-10	0.1848E-10	0.2468E-09	0.2483E-09
500	0.1000E 01	0.1664E-02	0.1665E-02	0.1850E-10	0.1906E-10	-0.6294E-09	-0.6373E-09
600	0.1200E 01	0.1662E-02	0.1662E-02	-0.2208E-09	-0.2210E-09	-0.4595E-08	-0.4604E-08
700	0.1400E 01	0.1668E-02	0.1689E-02	-0.6749E-09	-0.6913E-09	-0.5478E-09	-0.6178E-09
800	0.1600E 01	0.1772E-02	0.1771E-02	0.1750E-09	0.1697E-08	0.6304E-07	0.6221E-07
900	0.1800E 01	0.1551E-02	0.1557E-02	0.1263E-07	0.1285E-07	0.1216E-06	0.1239E-06
998	0.1996E 01	0.4369E-04	0.4317E-04	0.1238E-08	0.1254E-08	-0.6028E-06	-0.6107E-06

TABLE-2

A Comparison Between Exact and Field Solutions for 3a Long Cylinder (Problem-1)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.1667E-02	0.1667E-02	0.2035E-15	0.0000E 79	0.0000E 79	0.0000E 79
150	0.3000E 00	0.1667E-02	0.1667E-02	-0.9066E-15	-0.1057E-12	-0.1763E-13	-0.5684E-13
300	0.6000E 00	0.1667E-02	0.1667E-02	-0.3442E-15	-0.8184E-13	0.8666E-13	-0.9095E-12
450	0.9000E 00	0.1667E-02	0.1667E-02	0.4753E-13	0.9599E-13	0.3795E-12	0.1194E-11
600	0.1200E 01	0.1667E-02	0.1667E-02	-0.2533E-12	-0.2176E-13	-0.6758E-11	-0.6821E-11
750	0.1500E 01	0.1667E-02	0.1667E-02	-0.1031E-11	-0.8238E-12	0.1784E-10	0.1751E-10
900	0.1800E 01	0.1667E-02	0.1667E-02	0.1861E-10	0.1867E-10	0.2467E-09	0.2494E-09
1050	0.2100E 01	0.1662E-02	0.1663E-02	-0.4983E-10	-0.4973E-10	-0.2269E-08	-0.2277E-08
1200	0.2400E 01	0.1668E-02	0.1689E-02	-0.6749E-09	-0.6913E-09	-0.5478E-09	-0.6178E-09
1350	0.2700E 01	0.1757E-02	0.1757E-02	0.6263E-08	0.6401E-08	0.1164E-06	0.1177E-06
1498	0.2996E 01	0.4369E-04	0.4317E-04	0.1238E-08	0.1245E-08	-0.6028E-06	-0.6107E-06

TABLE-3

A Comparison Between Exact and Field Solutions for 4a Long Cylinder (Problem-1)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.1667E-02	0.1667E-02	0.3524E-18	0.4003E-13	0.0000E 79	0.0000E 79
200	0.4000E 00	0.1667E-02	0.1667E-02	-0.2747E-17	-0.1662E-12	-0.3721E-16	-0.1705E-12
400	0.8000E 00	0.1667E-02	0.1667E-02	0.3208E-16	0.5790E-13	0.6807E-15	0.4547E-12
600	0.1200E 01	0.1667E-02	0.1667E-02	-0.2475E-15	-0.3116E-12	-0.9225E-14	0.0000E-79
800	0.1600E 01	0.1667E-02	0.1667E-02	-0.3414E-15	-0.3116E-12	0.8663E-13	0.0000E-79
1000	0.2000E 01	0.1667E-02	0.1667E-02	0.5078E-13	0.2952E-12	-0.4380E-12	0.5684E-12
1200	0.2400E 01	0.1667E-02	0.1667E-02	-0.1181E-11	-0.9633E-12	-0.6858E-11	0.6992E-11
1400	0.2800E 01	0.1667E-02	0.1667E-02	0.1861E-10	0.1867E-10	0.2467E-09	0.2494E-09
1600	0.3200E 01	0.1662E-02	0.1663E-02	-0.2208E-09	-0.2210E-09	-0.4595E-08	-0.4604E-08
1800	0.3600E 01	0.1772E-02	0.1771E-02	0.1750E-08	0.1697E-08	0.6304E-07	0.6221E-07
1998	0.3996E 01	0.4369E-04	0.4317E-04	0.1238E-08	0.1254E-08	-0.6028E-06	-0.6107E-06

TABLE-4

A Comparison Between Exact and Field Solutions for 2a Long Cylinder (Problem-2)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.5462E-02	0.5396E-02	0.1589E-06	0.1609E-06	-0.4167E-05	-0.4167E-05
100	0.2000E 00	0.1801E-02	0.1825E-02	-0.3039E-07	-0.3040E-07	-0.2886E-06	-0.3108E-06
200	0.4000E 00	-0.1474E-03	-0.1433E-03	-0.1576E-07	-0.1607E-07	0.2628E-06	0.2635E-06
300	0.6000E 00	-0.1513E-03	-0.1489E-03	0.2517E-10	0.4362E-10	0.5739E-07	0.5692E-07
400	0.8000E 00	-0.1026E-04	-0.9448E-05	0.1148E-08	0.1151E-08	-0.1115E-07	-0.1125E-07
500	0.1000E 00	0.9576E-05	0.9467E-05	0.1573E-09	-0.1601E-09	-0.5715E-08	-0.5728E-08
600	0.1200E 00	0.2072E-05	0.2013E-05	-0.6169E-10	-0.6243E-10	0.1825E-10	0.3114E-10
700	0.1400E 00	-0.4085E-06	-0.4086E-06	-0.1998E-10	-0.1964E-10	0.4179E-09	0.4121E-09
800	0.1600E 00	-0.2064E-06	-0.2034E-06	0.1737E-11	0.1786E-11	0.5718E-10	0.5701E-10
900	0.1800E 00	-0.1037E-08	-0.9738E-08	0.1797E-11	0.2029E-11	-0.2189E-10	-0.2207E-10
998	0.1996E 00	0.6003E-09	0.4969E-09	0.2891E-13	0.3673E-13	-0.1447E-10	-0.1837E-10

TABLE-5

A Comparison Between Exact and Field Solutions for 3a Long Cylinder (Problem-2)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.5462E-02	0.5396E-02	0.1589E-06	0.1609E-06	-0.4167E-05	-0.4167E-05
150	0.3000E 00	0.4109E-03	0.4031E-03	-0.2910E-07	-0.2942E-07	0.2247E-06	0.2254E-06
300	0.6000E 00	-0.1513E-03	-0.1489E-03	0.2517E-10	0.4362E-10	0.5739E-07	0.5692E-07
450	0.9000E 00	0.8273E-05	0.8241E-05	0.5673E-09	0.5693E-09	-0.1059E-07	-0.1055E-07
600	0.1200E 00	0.2072E-05	0.2121E-05	-0.6169E-10	-0.6245E-10	-0.1829E-10	-0.1016E-10
750	0.1500E 00	-0.3856E-06	-0.3845E-06	-0.4460E-11	-0.4766E-11	0.2055E-09	0.2102E-09
900	0.1800E 00	0.2453E-08	0.3056E-08	0.1645E-11	0.1644E-11	-0.2249E-10	-0.2247E-10
1050	0.2100E 00	0.7446E-08	0.7478E-08	-0.9487E-13	-0.9458E-13	-0.1597E-11	-0.1663E-11
1200	0.2400E 00	-0.8213E-09	-0.8124E-09	-0.2282E-13	-0.2361E-13	0.6123E-12	0.6197E-12
1350	0.2700E 00	-0.2442E-10	-0.2296E-10	0.3552E-14	0.3962E-14	-0.2873E-13	-0.3909E-13
1498	0.2996E 00	-0.2442E-10	-0.1114E-11	-0.7104E-15	-0.5122E-16	-0.8288E-14	-0.2562E-13

TABLE-6

A Comparison Between Exact and Field Solutions for 4a Long Cylinder (Problem-2)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.5462E-02	0.5396E-02	0.1589E-06	0.1609E-06	-0.4167E-05	-0.4167E-05
200	0.4000E 00	-0.1474E-03	-0.1433E-03	-0.1576E-07	-0.1607E-07	0.2628E-06	0.2635E-06
400	0.8000E 00	-0.1026E-04	-0.9448E-05	0.1149E-08	0.1151E-08	-0.1115E-07	-0.1125E-07
600	0.1200E 01	0.1964E-05	0.1908E-05	-0.6161E-10	-0.6233E-10	0.5864E-10	0.7039E-10
800	0.1600E 01	-0.2072E-06	-0.2041E-06	0.1715E-11	0.1746E-11	0.5656E-10	0.5620E-10
1000	0.2000E 01	0.1520E-07	0.1505E-07	0.1095E-12	0.1128E-12	-0.7232E-11	-0.7273E-11
1200	0.2400E 01	-0.8224E-09	0.8130E-09	-0.2284E-13	-0.2360E-13	0.6136E-12	0.6195E-12
1400	0.2800E 01	-0.5189E-10	-0.1917E-10	0.1510E-14	0.2541E-14	0.2585E-20	0.4031E-13
1600	0.3200E 01	0.8303E-09	0.3573E-11	0.1510E-13	0.2018E-15	-0.6345E-19	0.1234E-14
1800	0.3600E 01	0.1495E-07	0.3083E-12	0.4832E-13	0.9502E-17	0.1026E-17	0.5638E-18
1998	0.3996E 01	-0.1328E-07	0.2190E-14	0.1160E-11	0.6774E-19	-0.1200E-16	-0.3388E-16

TABLE-7

A Comparison Between Exact and Field Solutions for 2a Long Cylinder (Problem-3)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.1667E-02	0.1667E-02	-0.6682E-13	-0.8528E-13	0.0000E 79	0.0000E 79
100	0.2000E 00	0.1667E-02	0.1667E-02	-0.4827E-12	-0.3543E-12	-0.7068E-11	-0.6878E-11
200	0.4000E 00	0.1667E-02	0.1667E-02	-0.5162E-12	-0.3823E-12	0.1707E-10	0.1648E-10
300	0.6000E 00	0.1667E-02	0.1667E-02	0.6095E-11	0.6217E-11	0.1275E-09	0.1264E-09
400	0.8000E 00	0.1666E-02	0.1666E-02	0.1882E-10	0.1935E-10	0.5591E-11	0.4263E-11
500	0.1000E 01	0.1664E-02	0.1664E-02	-0.4801E-10	-0.4928E-10	-0.1744E-08	-0.1766E-08
600	0.1200E 01	0.1670E-02	0.1670E-02	-0.3505E-09	-0.3622E-09	-0.3401E-08	-0.3414E-08
700	0.1400E 01	0.1713E-02	0.1713E-02	-0.7681E-11	-0.1321E-10	0.1751E-07	0.1758E-07
800	0.1600E 01	0.1712E-02	0.1714E-02	0.4809E-08	0.4804E-08	0.8019E-07	0.8048E-07
900	0.1800E 01	0.1117E-02	0.1103E-02	0.9274E-08	0.9392E-08	-0.8808E-07	-0.9601E-07
998	0.1996E 01	0.1126E-05	0.1099E-05	-0.4598E-07	-0.4716E-07	-0.1238E-05	-0.1254E-05

TABLE-8

A Comparison Between Exact and Field Solutions for 3a Long Cylinder (Problem-3)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.1667E-02	0.1667E-02	0.1131E-16	0.4030E-13	0.0000E 79	0.0000E 79
150	0.3000E 00	0.1667E-02	0.1667E-02	-0.1294E-14	-0.1266E-12	-0.1113E-13	-0.4547E-12
300	0.6000E 00	0.1667E-02	0.1667E-02	0.6975E-14	0.9825E-13	0.1872E-12	0.1194E-11
450	0.9000E 00	0.1667E-02	0.1667E-02	0.2895E-13	-0.6129E-13	-0.4872E-12	-0.2274E-12
600	0.1200E 01	0.1667E-02	0.1667E-02	-0.5155E-12	-0.2932E-12	-0.6874E-11	-0.6821E-11
750	0.1500E 01	0.1667E-02	0.1667E-02	0.1361E-11	0.1596E-11	0.6272E-10	0.6258E-10
900	0.1800E 01	0.1666E-02	0.1666E-02	0.1882E-10	0.1955E-10	0.5579E-11	0.3240E-11
1050	0.2100E 01	0.1664E-02	0.1664E-02	-0.1731E-09	-0.1756E-09	-0.3232E-08	-0.3260E-08
1200	0.2400E 01	0.1713E-02	0.1713E-02	-0.7681E-11	-0.1321E-10	0.1751E-07	0.1758E-07
1350	0.2700E 01	0.1541E-02	0.1543E-02	0.8879E-09	0.9089E-08	0,6857E-07	0.6962E-07
1498	0.2996E 01	0.1126E-05	0.1099E-05	-0.4598E-07	-0.4716E-07	-0.1238E-05	-0.1254E-05

TABLE-9

A Comparison Between Exact and Field Solutions for 4a Long Cylinder (Problem-3)

Point	Distance x	Deflection Y Exact	Deflection Y Field	Moment M Exact	Moment M Field	Shear Q Exact	Shear Q Field
0	0.0000E 79	0.1667E-02	0.1667E-02	0.1665E-18	0.4000E-13	0.0000E 79	0.0000E 79
200	0.4000E 00	0.1667E-02	0.1667E-02	-0.2829E-17	-0.1265E-12	0.2785E-17	0.0000E 79
400	0.8000E 00	0.1667E-02	0.1667E-02	0.5192E-16	0.6280E-13	0.5204E-15	0.2842E-12
600	0.1200E 01	0.1667E-02	0.1667E-02	0.7037E-15	0.3116E-12	-0.1196E-13	0.0000E 79
800	0.1600E 01	0.1667E-02	0.1667E-02	0.6978E-14	0.3143E-12	0.1872E-12	-0.5684E-13
1000	0.2000E 01	0.1667E-02	0.1667E-02	-0.3341E-13	-0.2054E-12	-0.2207E-12	-0.2258E-11
1200	0.2400E 01	0.1667E-02	0.1667E-02	-0.5232E-12	-0.2882E-12	0.1726E-10	0.1751E-10
1400	0.2800E 01	0.1668E-02	0.1666E-02	0.1882E-10	0.1954E-10	0.5579E-10	0.3240E-11
1600	0.3200E 01	0.1670E-02	0.1670E-02	-0.3505E-09	-0.3622E-09	-0.3401E-08	-0.3414E-08
1800	0.3600E 01	0.1712E-02	0.1714E-02	0.4809E-08	0.4804E-08	0.8019E-07	0.8048E-07
1998	0.3996E 01	0.1126E-05	0.1099E-05	-0.4598E-07	-0.4716E-07	-0.1238E-05	-0.1254E-05

VITA

Pravin R. Ghael, the son of Mr. and Mrs. Rangildas D. Ghael, was born on May 25, 1942, at Surat, Gujarat State, India. He graduated from N. M. Jhaveri Union High School, Surat, India in 1959. He attended two years of college at the M. T. B. College of Science, Surat, India. After that in March, 1963, he enrolled in the University of Tennessee at Knoxville, Tennessee and received his Bachelor of Science Degree in Mechanical Engineering in December, 1965.

After graduating, he was employed as a design engineer at the Jeffrey Manufacturing Company, Columbus, Ohio for one year.

In January, 1967, he enrolled in the University of Missouri at Rolla for work toward his Master's Degree in Mechanical Engineering.

He was married to Miss Usha H. Mandlaywala on December 11, 1967.